Improved approximation algorithms for minimum power covering problems

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Abstract

Given an undirected graph with edge costs, the **power of a node** is the maximum cost of an edge incident to it, and the **power of a graph** is the sum of the powers of its nodes. Motivated by applications in wireless networks, we consider two network design problems under the power minimization criteria. In both problems we are given a graph $G = (V, E)$ with edge costs and a set $T \subseteq V$ of terminals. The goal is to find a minimum power edge subset $F \subseteq E$ such that the graph $H = (V, F)$ satisfies some prescribed requirements. In the Min-Power Edge-Cover problem H should contain an edge incident to every terminal. Using the Iterative Randomized Rounding (IRR) method, we give an algorithm with expected approximation ratio 1.41; the ratio is reduced to $73/60 < 1.22$ when T is an independent set in G. In the case of unit costs we give a simple algorithm with ratio 5/4. For all these NP-hard problems the previous best known ratio was 3/2. In the related Min-Power Terminal Backup problem H should contain a path from every $t \in T$ to some node in $T \setminus \{t\}$. We obtain ratio 3/2 for this NP-hard problem, improving the trivial ratio of 2.

Keywords: Approximation algorithms; Iterative randomized rounding; Minimum power; Edge-cover; Terminal backup

1 Introduction

1.1 Motivation

Wireless networks are studied extensively due to their wide applications. The power consumption of a station determines its transmission range, and thus also the stations it can send messages to; the power typically increases at least quadratically in the transmission range. Assigning power levels to the stations (nodes) determines the resulting communication network. Conversely, given a communication network, the power required at v only depends on the farthest node that is reached directly by v. This is in contrast with wired networks, in which every pair of stations that need to communicate directly incurs a cost. Thus the power $p(v)$ of a node v is the cost of its largest cost edge touching v. The power of a graph is the sum of power of nodes. The first work under the minimum power model is from 1989 [6]. For a sample of other work under the minimum power model see for example [1, 3, 14, 20, 24, 8, 4, 5, 7, 9, 13, 16, 17, 19, 22, 23, 26, 27, 28, 21, 15].

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1.2 Basic definitions

Definition 1 *Let $H = (V, F)$ be a graph with edge-costs $\{c(e) : e \in F\}$. For $v \in V$, the* **power** $p(v) = p_H(v) = p_F(v)$ **of** *v in H (w.r.t. c) is the maximum cost of an edge in F incident to v (or zero, if no such edge exists), i.e., $p(v) = p_F(v) = \max_{vu \in F} c(vu)$. The power of H is the sum of the powers of its nodes, namely, $p(H) = p(F) = \sum_{v \in V} p_F(v)$.*

All the graphs are assumed to be undirected, unless stated otherwise. In our problems, the input is a graph $G = (V, E)$ with edge costs $\{c(e) : e \in E\}$ and a subset $T \subseteq V$ of terminals; the goal is to find a minimum power subgraph $H = (V, F)$ of G that satisfies some prescribed properties. We refer the reader to a recent survey [25] on such problems. We consider the min-power variant of two classic problems Edge-Cover and Terminal-Backup defined below.

Definition 2 *For a graph $H = (V, F)$ and a set $T \subseteq V$ of terminals, we say that F (or H) is:*

- *a T*-**cover** *if every $t \in T$ has some edge in F incident to it (equivalently, no connected component of H is a single terminal);*
- *a T*-**backup** *if every $t \in T$ has a path to some other node in T (equivalently, no connected component of H contains a single terminal).*

1.3 The problems considered

Min-Power Edge-Cover Here F should be a T-cover, namely, every $t \in T$ has some edge in F incident to it.

Min-Power Terminal Backup Here F should be a T-backup, namely, every $t \in T$ has a path to some other node in T.

In the case $T = V$ the problems coincid; the resulting min-power problem is still NP-hard with a standard reduction from Set Cover. The min-cost versions (where one seeks to minimize $c(F) = \sum_{e \in F} c(e)$) of these problems can be solved in polynomial time see the book [29]. Min-Cost Edge Cover can be considered among the the most basic problems in theoretical computer science. However, the Min-Power Edge-Cover problem is NP-hard even if T is an independent set in the input graph G and unit costs [14]. The NP-hardness proof in [14] easily extends to the Min-Power Terminal Backup problem.

For each of these problems, one can remove any edge from any cycle without destroying feasiblity or increasing the objective function, so any inclusion-minimal solution is a forest. It is known that if F is a forest then $c(F) \leq p(F) \leq 2c(F)$. This implies that both problems admit ratio 2, by simply computing an optimal min-cost solution.

For Min-Power Edge-Cover the trivial ratio 2 was improved to 1.5 in [18]. No better ratio was known even the case when T is an independent set in G and unit costs. We improve this as follows.

Theorem 3 Min-Power Edge-Cover *admits a polynomial time algorithm with expected approximation ratio* 1.41*. If T is an independent set in G then the ratio can be reduced to* 73/60*.*

The algorithm in Theorem 3 uses the Iterative Randomized Rounding (IRR) method. We also use a method of analysing the best of two algorithm using a convex combination of their results; we have seen this technique in [11]. In the case of unit costs we use a different method to obtain a better ratio.

Theorem 4 Min-Power Edge-Cover *with unit costs admits ratio* $5/4$.

We also improve the trivial ratio 2 for Min-Power Terminal Backup.

Theorem 5 Min-Power Terminal Backup *admits ratio* 1.5.

The proof of the latter theorem uses the idea of the 3/2-approximation algorithm in [18] for Min-Power Edge-Cover, but the details are more involved.

Work using the iterative randomized rounding technique: Byrka, Grandoni, Rothvoß and Sanita [2] give a $\ln 4 + \epsilon < 1.39$ approximation for the Min-Cost Steiner Tree problem, in which the goal is finding a tree that spans all terminals, with minimum sum of costs over the edges. This is the best ratio known for the problem, and the first one to give an approximation better than 2 with respect to an LP. Goemans, Olver, Rothvoß and Zenklusen [10] uses matroids technique to get a faster and simpler $\ln 4 + \epsilon$ approximation for the same problem. The main difference is that [10] updates the LP greedily, using matroid techniques. In addition [10] gives a better ratio for quasi-bipartite graphs. Grandoni [12] gives the best known 1.91 ratio for the Min-Power Steiner Tree problem. Our paper has similarities with [12] including a Harmonic potential function, and two main differences: it is technically easier (for us) to cover terminals than to cover all cuts separating terminals as in [12], and we have to combine iterative randomized rounding with another algorithm since by itself, iterative randomized rounding fails to improve the existing ratio in some cases.

2 Algorithm for Min-Power Edge-Cover (Theorem 3)

A **star** is a rooted tree R such that only its root r, called the **center**, may have degree ≥ 2. For Min-Power Edge-Cover, any inclusion-minimal solution is a collection of disjoint stars, as if any path in a solution has length three, the middle edge of this path can be removed from the solution without violating feasibility.

For $S \subseteq T$ let π_S be the minimum power of a star R_S that contains S. Note that given S, both R_S and π_S can be computed in polynomial time by "guessing" the center of R_S. For an integer $k \geq 1$ let $\mathcal{T}_k = \{S \subseteq T : |S| \leq k\}$. We say that a subfamily $\mathcal{T} \subseteq \mathcal{T}_k$ is a **k-restricted T-cover** if the union of the sets in $\mathcal{T}$ is T; the power of $\mathcal{T}$ is defined to be $p(\mathcal{T}) = \sum_{S \in \mathcal{T}} \pi_S$. In what follows we denote by $t = |T|$ the number of terminals.

Lemma 6 Min-Power Edge-Cover *with $t = |T|$ terminals can be solved optimally in time $2^{t \log_2 t} poly(n)$.*

Proof. For $S \subseteq T$ let R_S be some minimum power star that contains S. For every partition $\mathcal{P}$ of T compute a solution $F_\mathcal{P} = \cup_{S \in \mathcal{P}} R_S$ for this partition, and among the solutions $F_\mathcal{P}$ computed return one of minimum power. Note that $F_\mathcal{P}$ can be computed in polynomial time for any $\mathcal{P}$, since R_S can be computed in polynomial time for each $S \in P$. For the partition $\mathcal{P}$ defined by the stars of some optimal solution, $F_{\mathcal{P}}$ is an overall optimal solution for the problem. The number of partitions of a set of size t is the Bell number B_t, and it is known that $B_t \leq 2^{t \log_2 t}$. The lemma follows. ■

The "hypergraphic" linear program $LP_k(T)$ below has a variable x_S for every $S \in \mathcal{T}_k$, and it is a relaxation for the problem of finding a k-restricted T-cover of minimum power.

$$
\begin{array}{lll}
\min & \displaystyle\sum_{S \in \mathcal{T}_k} \pi_S x_S & \\
\text{s.t.} & \displaystyle\sum_{S \in \mathcal{T}_k, S \ni v} x_S \geq 1 & \forall v \in T \\
 & x_S \geq 0 & \forall S \in \mathcal{T}_k
\end{array}
$$

By Lemma 6, $LP_k(T)$ can be solved in polynomial time for any fixed k.

Let us call a feasible solution x to $LP_k(T)$ **irreducible** if no coordinate of x can be lowered while keeping feasibility. We use n to denote $|V(G)|$.

Lemma 7 *Let x be an irreducible solution to $LP_k(T)$. Then $\sum_{S \in \mathcal{T}_k} x_S \leq n$, and $\mathcal{T}_k$ with probabilities $Pr[S] = x_S/n$ for $S \neq \emptyset$ and $Pr[\emptyset] = 1 - \sum_{S \in \mathcal{T}_k} x_S/n$ is a sample space, in which $Pr[\{S \in \mathcal{T}_k : S \ni v\}] \geq 1/n$ holds for any $v \in T$.*

Proof. Since x is irreducible, for any $S \in \mathcal{T}_k$ with $x_S > 0$ there exists $v \in S$ such that the inequality of v in $LP_k(T)$ is tight. For every $S \in \mathcal{T}_k$ with $x_S > 0$ choose one such node v_S. Let $W = \{v_S : x_S > 0, S \in \mathcal{T}_k\}$ be the set of chosen nodes, and note that $W \subseteq T$. Then

$$
\sum_{S \in \mathcal{T}_k} x_S \leq \sum_{v \in W} \sum_{S \in \mathcal{T}_k, S \ni v} x_S \leq \sum_{v \in W} 1 \leq |W| \leq n \ .
$$

This implies that $Pr[\emptyset] = 1 - \sum_{S \in \mathcal{T}_k} x_S/n \geq 0$ and thus we have a sample space. Furthermore, $Pr[\{S \in \mathcal{T}_k : S \ni v\}] = \sum_{S \in \mathcal{T}_k, S \ni v} x_S/t \geq 1/t$, by the constraint of v in $LP_k(T)$. ■

The following lemma provides a (tight) bound on the ratio between the powers of an optimal T-cover and a k-restricted T-cover.

Lemma 8 (Kortsarz & Nutov [18]) *For any T-cover F there exists a k-restricted T-cover $\mathcal{T}$ of power $p(\mathcal{T}) \leq (1 + 1/k)p(F)$.*

We run two algorithms and take the best of the two. The first algorithm is the 3/2-approximation algorithm of Kortsarz & Nutov [18]; we call it the KN-Algorithm.

Algorithm 1: KN-Algorithm$(G = (V, E), c, T)$

for all $u, v \in T$ (possibly $u = v$) compute a min-power $\{u, v\}$-edge-cover J_{uv}
let (T, E') be a complete graph with all loops and edge costs $c_{uv} = p(J_{uv})$ for all $u, v \in T$
compute a minimum cost T-edge-cover $J' \subseteq E'$
return $J = \displaystyle\bigcup_{uv \in J'} J_{uv}$

The second algorithm is an Iterative Randomized Rounding algorithm, abbreviated by IRR-Algorithm. For previous applications of this type of algorithms see [2] for the Min-Cost Steiner Tree problem, and [12] for the Min-Power Steiner Tree problem.

Algorithm 2: IRR-ALGORITHM$(G=(V,E),c,T,k)$

initialize $J \leftarrow \emptyset$
while $T \neq \emptyset$ **do**
  compute an irreducible optimal solution x for $LP_k(T)$ and sample one star $S \in \mathcal{T}_k$ with probabilities as in Lemma 7
  $T \leftarrow T \setminus S$, $J \leftarrow J \cup R_S$
return J

Note that in every iteration, the set of terminals may change. In such a case, the IRR-Algorithm solves a new LP with respect to the new set of terminals. To ensure polynomial time, after $2n \ln n$ iterations the while-loop is terminated, and we add to J a solution for the residual problem computed by the KN-Algorithm. The following lemma shows that the expected loss in the approximation ratio incurred by such modification is negligible.

Lemma 9 *In every iteration, every $v \in T$ is hit with probability at least $1/n$. The probability that $T \neq \emptyset$ after $2n \ln n$ iterations is at most $1/n$. The expected loss in the approximation ratio incurred by stopping the IRR algorithm after $2n \ln n$ iterations is at most $\frac{3}{2n}$.*

Proof. The first statement follows Lemma 7. The probability that after $i = 2n \ln n$ iterations an element is not hit is at most $(1-1/n)^i \leq 1/n^2$. By the union bound the probability that there exists a non hit terminal is at most $1/n$. Finally, in the case that there exists a non hit terminal, the algorithm has an approximation ratio of 3/2. Thus the loss in the approximation ratio is at most $\frac{3}{2n}$. ■

We now give properties of these algorithms that will enable us to prove the approximation ratio. We say that a star R is a **proper star** if R has at least one terminal and if R has at least two edges then all the leaves of R are terminals. Fix some proper star R with center r. Note that if R has a single edge then r can be the unique terminal in R. Denote the leaves of R by $v_1, v_2, \ldots, v_q$ arranged by non-increasing edge costs $c_1 \geq c_2 \geq \ldots \geq c_q$ where $c_j = c(rv_j)$ and assume that $c_1 > 0$. Note that $p(R) = c_1 + c(R) = c_1 + \sum_{j=1}^{q} c_j$. Let $\psi(R)$ be defined by:

$$\psi(R) = \begin{cases} c_3 + c_5 + \cdots + c_q & q \geq 3 \text{ odd} \\ c_3 + c_5 + \ldots + c_{q-1} & q \geq 4 \text{ even}, \ r \notin T \\ c_3 + c_5 + \ldots + c_{q-1} + c_q & q \geq 4 \text{ even}, \ r \in T \end{cases}$$

Here $\psi(R) = 0$ if $q \in \{1,2\}$, except that $\psi(R) = c_2$ if $q = 2$ and $r \in T$.

The following lemma is proved in [18], but we provide a proof-sketch for completeness of exposition.

Lemma 10 ([18]) *Let R be a proper star as above. Then there exists a 2-restricted cover $\mathcal{T}$ of the terminals in R such that $p(\mathcal{T}) \leq p(R) + \psi(R) \leq \frac{3}{2}p(R)$.*

Proof. It is not hard to verify that the following $\mathcal{T}$ is as required:

$$\begin{array}{rcll} \mathcal{T} & = & \{\{v_1,v_2\},\{v_3,v_4\},\ldots,\{v_{q-2},v_{q-1}\},\{v_q\}\} & q \text{ odd}, \ \ r \notin T \\ \mathcal{T} & = & \{\{v_1,v_2\},\{v_3,v_4\},\ldots,\{v_{q-2},v_{q-1}\},\{v_q,r\}\} & q \text{ odd}, \ \ r \in T \\ \mathcal{T} & = & \{\{v_1,v_2\},\{v_3,v_4\},\ldots,\{v_{q-3},v_{q-2}\},\{v_{q-1},v_q\}\} & q \text{ even}, \ \ r \notin T \\ \mathcal{T} & = & \{\{v_1,v_2\},\{v_3,v_4\},\ldots,\{v_{q-3},v_{q-2}\},\{v_{q-1}\},\{v_q,r\}\} & q \text{ even}, \ \ r \in T \end{array}$$

It is also not hard to see that $\psi(R) \le \frac{1}{2}p(R)$. ■

Assume for a moment that R as above contains all terminals and is an optimal solution to our problem. Then $p(R)+\psi(R)$ bounds the solution value produced by the KN-Algorithm. We will show later that the expected solution value produced by the IRR-Algorithm is bounded by $p(R)+\phi(R)$ where

$$\phi(R) = \begin{cases} \sum_{j=1}^{q} c_j/j & q \ge 1, V(R) \subseteq T \\ \sum_{j=2}^{q} c_j/j & q \ge 1, r \notin T \\ 0 & \text{otherwise } (q=1,\ V(R) \cap T = \{r\}) \end{cases}$$

The function $\phi(R)$ is built so that the proof of Lemma 12 holds; we note that Harmonic functions are also used in [12] and [10].

If we know that our optimal solution is just one star R, then by taking the best outcome of the two algorithms, the (expected) value of the produced solution will be $p(R)+\min\{\psi(R), \phi(R)\}$. In the case of many stars, we take a convex combination of the two algorithms: KN-Algorithm with probability $\theta = 2/3$ and IRR-Algorithm with probability $1-\theta = 1/3$. Since any inclusion-minimal solution is a collection of node-disjoint proper stars, we conclude that the (expected) approximation ratio of the convex combination algorithm is bounded by the maximum possible value of

$$\frac{\theta(p(R)+\psi(R)) + (1-\theta)(p(R)+\phi(R))}{p(R)} = 1 + \frac{1}{3} \cdot \frac{2\psi(R)+\phi(R)}{p(R)}$$

over all the stars R (this assumes that, as shown in Lemma 13 below, the expected power of the output of the IRR-Algorithm is $p(R)+\phi(R)$).

For a proper star R as above let us denote (with some abuse of notation) $p(q) = p(R)$, $\psi(q) = \psi(R)$, and $\phi(q) = \phi(R)$. Then the expected approximation ratio of the convex combination algorithm is bounded by $\max_{q \ge 1} \rho(q)$, where

$$\rho(q) = 1 + \frac{1}{3} \max_{c_1 \ge \cdots \ge c_q \ge 0, c_1 > 0} \frac{2\psi(q)+\phi(q)}{p(q)} .$$

We will show later that:

Lemma 11 $\rho(q) \le 1\frac{73}{180} < 1.4056$.

Let $\Phi(R) = p(R)+\phi(R)$ and $\Psi(R) = p(R)+\psi(R)$. It is convenient to also have $\Phi(R) = 0$ if the star R has only a center and no leaves (this is not a proper star, and has $p(R) = 0$). The next lemma, to be proved in Section 3, is the heart of the proof of Theorem 3.

Lemma 12 *Consider an iteration of the IRR-Algorithm. Let R be a proper star at the beginning of the iteration and let R' be a star obtained from R by removing all the leaves of R that are terminals covered at the iteration, with one exception: if only the center of R is an uncovered terminal among $V(R)$ after the iteration, we keep in R' from R the leaf closest to the center (this means that, unless all the terminal vertices of R are covered, R' remains a proper star). Then $\Phi(R) - E[\Phi(R')] \ge p(R)/n$.*

For a collection $\mathcal{R}$ of stars let $\Phi(\mathcal{R}) := \sum_{R\in\mathcal{R}} \Phi(R)$ and $p(\mathcal{R}) := \sum_{R\in\mathcal{R}} p(R)$.

Lemma 13 *Let $\mathcal{R} = \{R_S : S \in \mathcal{T}\}$ be a set of stars of a k-restricted (optimal) T-cover $\mathcal{T}$ and J a solution produced by the IRR-Algorithm. Then $E[p(J)] \le \Phi(\mathcal{R})$.*

Proof. Let T_{i-1} be the set of terminals uncovered at the beginning of iteration i and τ_i^* the expected optimal value of $LP_k(T_{i-1})$. Let $\mathcal{R}_0 = \mathcal{R}$ and for $i \ge 1$ obtain $\mathcal{R}_i$ from $\mathcal{R}_{i-1}$ by taking, for each proper star in $R \in \mathcal{R}_{i-1}$, the star R' as in Lemma 12.

Now, note the following:

- $E[p(J)] \le \sum_{i\ge1} \tau_i^*/n$, since after solving $LP_k(T_{i-1})$ at iteration $i \ge 1$, each star R_S is selected with probability x_S/n.
- $\tau_i^* \le E[p(\mathcal{R}_{i-1})]$ at iteration $i \ge 1$, since the stars in $\mathcal{R}_{i-1}$ cover T_{i-1} while τ_i^* is the expected optimal value of $LP_k(T_{i-1})$.
- $E[p(\mathcal{R}_{i-1})]/n \le E[\Phi(\mathcal{R}_{i-1}) - \Phi(\mathcal{R}_i)]$ at iteration $i \ge 1$, by Lemma 12.

Combining we get that the expected power of J is bounded by:

$$E[p(J)] \le \sum_{i\ge1} \tau_i^*/n \le \sum_{i\ge1} E[p(\mathcal{R}_{i-1})]/n \le \sum_{i\ge1} E[\Phi(\mathcal{R}_{i-1}) - \Phi(\mathcal{R}_i)] = \Phi(\mathcal{R})$$

The last equality holds since the sum is telescopic and since $\Phi(\mathcal{R}_0) = \Phi(\mathcal{R})$ is not a random variable. ■

Let J_{KN} and J_{IRR} be the outputs of the KN-Algorithm and the IRR-Algorithm, respectively. Let $\mathcal{R}$ and $\mathcal{R}_k$ be optimal and k-restricted optimal set of stars that cover T, respectively. Then $p(\mathcal{R}_k) \le (1+1/k)p(\mathcal{R})$, by lemma 8. As was mentioned, in [18] it is proved that $p(J_{KN}) \le \Psi(\mathcal{R})$. By Lemma 13, $p(J_{IRR}) \le \Phi(\mathcal{R})$. Combining we get that the power of the solution produced by the convex combination of the two algorithms is bounded by

$$\theta p(J_{KN}) + (1-\theta)p(J_{IRR}) \le \theta\Psi(\mathcal{R}) + (1-\theta)\Phi(\mathcal{R}_k) \le \left(1+\frac{1}{k}\right)(\theta\Psi(\mathcal{R}) + (1-\theta)\Phi(\mathcal{R}))$$

From Lemma 11 we conclude that $\theta\Psi(\mathcal{R}) + (1-\theta)\Phi(\mathcal{R}) \le 1.4056p(\mathcal{R})$ for $\theta = 2/3$. Consequently, we get that for $\theta = 2/3$ and constant k large enough

$$\theta p(J_{KN}) + (1-\theta)p(J_{IRR}) \le 1.41p(\mathcal{R}) = 1.41 \cdot \mathsf{opt}\ .$$

To complete the proof of the 1.41 approximation ratio it only remains to prove Lemmas 11 and 12; Lemma 11 is proved below, while Lemma 12 is proved in the next section.

For the proof of Lemma 11 we bound the function $h(q) = 3(\rho(q) - 1)$, so $\rho(q) = 1 + \frac{1}{3}h(q)$. For simplicity of notation let us write

$$h(q) = \frac{2\psi(q) + \phi(q)}{p(q)} \quad \text{meaning} \quad h(q) = \max_{c_1\ge c_2\ge\cdots\ge c_q, c_1>0} \frac{2\psi(q) + \phi(q)}{p(q)}\ .$$

Note that Lemma 11 follows immediately from the following lemma:

Lemma 14 $h(q) \le \frac{73}{60}$.

Proof. Let us consider the cases $q = 1, 2, 3, 4, 5$.

1. $\psi(1) = 0$, $\phi(1) \le c_1$ and $p(1) = 2c_1 > 0$, hence $h(1) = 1/2$.

2. $\psi(2) = c_2$, $\phi(2) = c_1 + c_2/2$, and $p(2) = 2c_1 + c_2$, hence $h(2) \le \frac{c_1+\frac{5}{2}c_2}{2c_1+c_2} \le \frac{7}{6}$.

3. We have $h(3) = \frac{2c_3+c_1+c_2/2+c_3/3}{2c_1+c_2+c_3} \le \frac{23}{24}$.

4. We have $h(4) = \frac{2c_3+2c_4+c_1+c_2/2+c_3/3+c_4/4}{2c_1+c_2+c_3+c_4}$, and this can be verified to be at most $\frac{73}{60}$ by expanding and using $c_1 \ge c_2 \ge c_3 \ge c_4$ (we get equality when $c_1 = c_2 = c_3 = c_4$).

5. We have $h(5) = \frac{2c_3+2c_5+c_1+c_2/2+c_3/3+c_4/4c_5/5}{2c_1+c_2+c_3+c_4+c_5}$, and this can be verified to be at most $\frac{73}{60}$ by expanding and using $c_1 \ge c_2 \ge c_3 \ge c_4 \ge c_5$.

For $q > 5$, we use induction on q. For even $q > 4$, we must prove: $60(2(c_3 + c_5 + \cdots c_{q-1} + c_q) + \sum_{j=1}^{q} c_j/j) \le 73(c_1 + \sum_{j=1}^{q} c_j)$, which follows from summing up the inductive hypothesis: $60(2(c_3 + c_5 + \cdots c_{q-3} + c_{q-2}) + \sum_{j=1}^{q-2} c_j/j) \le 73(c_1 + \sum_{j=1}^{q-2} c_j)$ and the inequalities $60c_q \le 60c_{q-2}$, $60c_{q-1} \le 60c_{q-2}$, $60c_q(1 + 1/q) \le 73c_q$, and $60c_{q-1}(1 + 1/(q-1)) \le 73c_{q-1}$.

For odd $q > 5$, we must prove: $60(2(c_3 + c_5 + \cdots + c_q) + \sum_{j=1}^{q} c_j/j) \le 73(c_1 + \sum_{j=1}^{q} c_j)$, which follows from summing up the inductive hypothesis: $60(2(c_3 + c_5 + \cdots + c_{q-2}) + \sum_{j=1}^{q-2} c_j/j) \le 73(c_1 + \sum_{j=1}^{q-2} c_j)$ and $(120 + 60/q)c_q + c_{q-1}60/(q-1) \le 73(c_{q-1} + c_q)$. ■

In the case when T is an independent set in G, no star has head in T. In this case, we simply the IRR-Algorithm (the improvement one gets from using a convex combination is minor). The approximation ratio stated for this case in Theorem 3 follows from the following lemma.

Lemma 15 *If T is an independent set in G, $\Phi(q)/p(q) \le \frac{73}{60}$.*

Proof. In this case, we have $\Phi(q) = \sum_{j=1}^{q} c_j(1 + 1/j)$. One obtains that $60\Phi(q) \le 73p(q)$ for $q = 1, 2, 3, 4$ by inspection, using $c_1 \ge c_2 \ge c_3 \ge c_4$. The bound is tight for $q = 4$ and $c_1 = c_2 = c_3 = c_4$. For $j \ge 5$, the bound follows from the fact that $60(1 + 1/j) \le 73$. ■

3 Proof of Lemma 12

Let us write explicitly the function Φ:

$$\Phi(R) = p(R) + \phi(R) = \begin{cases} c_1 + \displaystyle\sum_{j=1}^{q} c_j(1 + 1/j) & q \ge 1, V(R) \subseteq T \\ \displaystyle\sum_{j=1}^{q} c_j(1 + 1/j) & q \ge 1, r \notin T \\ 2c_1 & \text{otherwise } (q = 1,\ V(R) \cap T = \{r\}) \end{cases}$$

We split the proof into two cases: $r \notin T$ and $r \in T$.

3.1 The case $r \notin T$

Recall that a set-function f on a groundset U is submodular if for any $A \subseteq U$ and $a_j, a_k \in U \setminus A$ we have:

$$\Delta_f(A, \{a_j, a_k\}) := f(A \cup \{a_j\}) + f(A \cup \{a_k\}) - f(A) - f(A \cup \{a_j, a_k\}) \geq 0 \ .$$

We will need the following lemma. We believe this lemma is known, but we failed to find its proof in the literature.

Lemma 16 *Let U be a set of items with non-negative weights $\{w(u) : u \in U\}$ and let $z_1 \geq z_2 \geq \cdots \geq z_{|U|}$ be reals. Let $f(\emptyset) := 0$ and for $\emptyset \neq A \subseteq U$ define $f(A) := \sum_{i=1}^{|A|} z_i w(a_i)$, where $a_1, \ldots, a_{|A|}$ is an ordering of A such that $w(a_1) \geq \cdots \geq w\left(a_{|A|}\right)$. Then f is submodular and non-decreasing.*

Proof. Let $A \subseteq U$ and $a_j, a_k \in U \setminus A$. Order the elements in $A \cup \{a_j, a_k\}$ in non-increasing order $a_1, \ldots, a_{|A|+2}$ by the weights $w_1 \geq \cdots \geq w_{|A|+2}$, and suppose w.l.o.g. that this order is $a_1, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_{k-1}, a_k, a_{k+1}, \ldots, a_{|A|+2}$. Note that the terms in the sums defining $f(A \cup \{a_k\})$ and $f(A)$ coincide up to the kth term, and this so also for $f(A \cup \{a_j\})$ and $f(A \cup \{a_j, a_k\})$. Then we have:

$$f(A \cup \{a_k\}) - f(A) = \sum_{i=k}^{|A|+2} w_i z_{i-1} - \sum_{i=k+1}^{|A|+2} w_i z_{i-2} = \sum_{i=k}^{|A|+2} w_i z_{i-1} - \sum_{i=k}^{|A|+1} w_{i+1} z_{i-1}$$

$$f(A \cup \{a_j\}) - f(A \cup \{a_j, a_k\}) = \sum_{i=k+1}^{|A|+2} w_i z_{i-1} - \sum_{i=k}^{|A|+2} w_i z_i = \sum_{i=k}^{|A|+1} w_{i+1} z_i - \sum_{i=k}^{|A|+2} w_i z_i$$

Consequently,

$$\begin{aligned}
\Delta_f(A, \{a_j, a_k\}) &= \sum_{i=k}^{|A|+2} w_i z_{i-1} - \sum_{i=k}^{|A|+1} w_{i+1} z_{i-1} + \sum_{i=k}^{|A|+1} w_{i+1} z_i - \sum_{i=k}^{|A|+2} w_i z_i \\
&= \sum_{i=k}^{|A|+2} w_i (z_{i-1} - z_i) - \sum_{i=k}^{|A|+1} w_{i+1} (z_{i-1} - z_i) \\
&\geq \sum_{i=k}^{|A|+1} (w_i - w_{i+1})(z_{i-1} - z_i) \geq 0
\end{aligned}$$

This shows that f is submodular. It is easy to see that f is non-decreasing. ∎

We want to show that $\Phi(R) - E[\Phi(R')] \geq p(R)/n$. Let $\bar{R}$ be the set of leaves of R. The case $\bar{R} = \emptyset$ is obvious hence we assume that $\bar{R} \neq \emptyset$.

In this case, by definition, $\Phi(R) = \sum_{j=1}^{q} (1 + 1/j) c_j$. Therefore if we set in Lemma 16 $w_i = c_i$ for every i and and $z_i = 1 + 1/i$ for $1 \leq i \leq q$, then by definition $f(\bar{R}) = \Phi(R)$.

Definition 17 *Let $\tilde{H}$ be the random variable of terminals hit in iteration i. For $H \subseteq T$ we denote the probability that $\tilde{H} \cap \bar{R} = H$ by $Pr[H]$, namely, that H is exactly the set of hit terminals among the vertices of R.*

Denote $\Delta(H) = \Phi(R) - \Phi(R')$; in this case ($r \notin T$), we have $\Delta(H) = f(\bar{R}) - f(\bar{R} \setminus H)$. Consider some arbitrary set $H =\subseteq \bar{R}$ of possible terminals that could be hit. This lemma is a standard consequence of submodularity:

Lemma 18 $\Delta(H) \geq \sum_{v \in H} \Delta(\{v\})$.

Proof.

$$\begin{aligned}
\Delta(H) &= f(\bar{R}) - f(\bar{R} \setminus H) = && \text{(As the sum is telescopic)} \\
&= \sum_{\ell=1}^{p} f\left(\bar{R} \setminus \{v_1, \ldots v_{\ell-1}\}\right) - f(\bar{R} \setminus \{v_1, \ldots v_\ell\}) \geq && \text{(As } f \text{ is submodular)} \\
&\geq \sum_{\ell=1}^{p} f(\bar{R}) - f(\bar{R} \setminus \{v_\ell\}) = \sum_{\ell=1}^{p} \Delta(\{v_\ell\})
\end{aligned}$$

■

Therefore

$$\begin{aligned}
E[\Delta(R)] &= \sum_{H \subseteq \bar{R}} Pr[H]\Delta(H) \geq && \text{(As } \Delta(H) \geq \sum_{\ell} \Delta(v_\ell)) \\
&\geq \sum_{H \subseteq \bar{R}} \left(Pr[H] \sum_{v \in H} \Delta(\{v\}) \right) = && \text{(By changing summation order)} \\
&= \sum_{v \in \bar{R}} \left(\Delta(\{v\}) \sum_{H \subseteq \bar{R} \mid v \in H} Pr[H] \right) \\
&= \sum_{v \in \bar{R}} \Delta(v) Pr[v \text{ is hit}] \geq \sum_{v \in \bar{R}} \Delta(v) \frac{1}{n}
\end{aligned}$$

To justify the last equality, note that $\sum_{H \subseteq \bar{R} \mid v \in H} Pr[H] = Pr[v \text{ is hit}]$ because we sum the probabilities of all sets H that contain v. The last inequality follows from Lemma 7.

What remains to be proved is that

$$\sum_{v \in \bar{R}} \Delta(v) \geq p(R) \tag{1}$$

We need to measure the change in the potential $\Delta(v_\ell)$ (recall that v_ℓ is the ℓ^{th} child of the star R). Also recall that in the potential function $\Phi(R)$, c_ℓ is multiplied by $(1 + 1/\ell)$. The addition of v_1 (and its most expensive edge) shifts all indexes by 1. This means that $v_{\ell-1}$ becomes v_ℓ. In the new star with v_1 the coefficient of the edge number ℓ is $1 + 1/\ell$ and in the star without this edges it was $1/(\ell - 1)$. Thus the difference between the coefficients is $-(1/(\ell - 1) - 1/\ell)$.

Suppose that we add rv_p, $p \geq 2$. Then the coefficients are shifted only for edges that are $p + 1$ smallest or later. This mean that the sum will start with $\ell = p + 1$. Indeed adding edge number p does not change the location of the $p - 1$ first edges. Thus the changes are as follows:

$$
\begin{aligned}
\Delta(\{v_1\}) &\geq 2c_1 - \sum_{\ell=2}^{q} c_\ell \left(\frac{1}{\ell-1} - \frac{1}{\ell}\right) \\
\Delta(\{v_2\}) &= c_2\left(1+\frac{1}{2}\right) - \sum_{\ell=3}^{q} c_\ell \left(\frac{1}{\ell-1} - \frac{1}{\ell}\right) \\
&\dots \\
\Delta(\{v_k\}) &= c_k\left(1+\frac{1}{k}\right) - \sum_{\ell=k+1}^{q} c_\ell \left(\frac{1}{\ell-1} - \frac{1}{\ell}\right) \\
&\dots \\
\Delta(\{v_q\}) &= c_q\left(1+\frac{1}{q}\right)
\end{aligned}
$$

Note that the coefficient of edge k is counted $k-1$ times and thus we get by summing up these equations that:

$$
\sum_{k=1}^{q} \Delta(v_k) \geq 2c_1 + \sum_{k}^{q} c_k \left(1 + \frac{1}{k} - (k-1)\left(\frac{1}{k-1} - \frac{1}{k}\right)\right) = 2c_1 + \sum_{k=2}^{q} c_k = p(R),
$$

ending the proof for the case $r \notin T$.

3.2 The case $r \in T$

Note that the equality $\Phi(R) - \Phi(R') = f(\bar{R}) - f(\bar{R}')$ no longer holds in all the cases, because $\Phi(R) = f(\bar{R}) + c_1$, but this may not hold for $\Phi(R')$. Precisely, the bound $\Delta(H)$ is by definition:

$$
\begin{aligned}
&\Delta(H) = f(\bar{R}) - f(\bar{R}') + c_1 \quad \text{if} \quad r \quad \text{is hit } (r \in H) \\
&\Delta(H) = f(\bar{R}) - f(\bar{R}') + c_1 - c_1' \quad \text{if} \quad r \quad \text{is not hit } (r \notin H) \text{ and } \bar{R} \neq H \\
&\Delta(H) = f(\bar{R}) + c_1 - 2c_q \quad \text{if} \quad \bar{R} = H
\end{aligned}
$$

Indeed, if r is hit, then c_1 does not appear anymore in $\phi(R')$, since r is no longer a terminal. If r is not hit, its power goes from c_1 to c_1'. In the case $\bar{R} = H$ we get that $\Phi(R) - \Phi(R') = (c_1 - 2c_q) + \sum_{j\geq 1}(1 + 1/j) \cdot c_j$. This is because R' is defined to keep from R only the leaf closest to the center, and therefore $\Phi(R') = 2c_q$.

Corollary 19 *If $\bar{R} \neq H$, we get that $\Phi(R) - \Phi(R') \geq f(\bar{R}) - f(\bar{R}')$. If $\bar{R} = H$, we get that $\Delta(H) = f(\bar{R}) + c_1 - 2c_q \geq f(\bar{R}) - c_1$.*

We continue with the proof of Lemma 12. We first assume that $Pr[\bar{R}] \leq 1/n$. Recall that

$$\begin{aligned}
E[\Delta(H)] &= Pr[\bar{R}]\cdot\Delta(\bar{R}) + \sum_{H\neq\bar{R}} Pr[H]\cdot\Delta(H) \quad \text{(Corollary 19 and } Pr[\bar{R}] \leq 1/n)\\
&\geq -\frac{1}{n}c_1 + Pr[\bar{R}]f(\bar{R}) + \sum_{H\neq\bar{R}} Pr[H]\Delta(H) \quad \text{(By separating } r \text{ from the sum)}\\
&\geq -\frac{1}{n}c_1 + Pr[\bar{R}]\Delta(\bar{R}) + \sum_{H \mid \ r\notin H \text{ and } H\neq\bar{R}} Pr[H]\Delta(H) + \sum_{H \mid \ r\in H} Pr[H]\Delta(H)\\
&\quad \text{(By the definition of } \Delta)\\
&\geq -\frac{1}{n}c_1 + Pr[\bar{R}]f(\bar{R}) + \sum_{H \mid \ r\notin H \text{ and } H\neq\bar{R}} Pr[H](f(\bar{R}) - f(\bar{R}\setminus H))\\
&\quad + \sum_{H \mid \ r\in H} Pr[H](c_1 + f(\bar{R}) - f(\bar{R}\setminus H))\\
&= -\frac{1}{n}c_1 + \sum_{H \mid \ r\notin H} Pr[H](f(\bar{R}) - f(\bar{R}\setminus H))\\
&\quad + \sum_{H \mid \ r\in H} Pr[H](c_1 + f(\bar{R}) - f(\bar{R}\setminus H)) \quad \text{(As in Lemma 18, submodularity)}\\
&\geq -\frac{1}{n}c_1 + \sum_{H \mid \ r\notin H}\sum_{v\in H} Pr[H]\sum_{v\in H}(f(\bar{R}) - f(\bar{R}\setminus\{v\}))\\
&\quad + \sum_{H \mid \ r\in H} Pr[H](c_1 + \sum_{v\in H\setminus\{r\}}(f(\bar{R}) - f(\bar{R}\setminus\{v\}))\\
&\quad \text{(By rearranging the terms)}\\
&= -\frac{1}{n}c_1 + \left(\sum_{v\in\bar{R}}(f(\bar{R}) - f(\bar{R}\setminus\{v\}))\cdot\sum_{H \mid v\in H} Pr[H]\right) + c_1\cdot\sum_{H \mid r\in H} Pr[H]\\
&\quad \text{(Lemma 9)}\\
&\geq -\frac{1}{n}c_1 + \left(\sum_{v\in\bar{R}}(f(\bar{R}) - f(\bar{R}\setminus\{v\}))\frac{1}{n}\right) + c_1\cdot\frac{1}{n}\\
&= \frac{1}{n}\sum_{v\in\bar{R}}(f(\bar{R}) - f(\bar{R}\setminus\{v\}))\\
&\geq p(R)/n,
\end{aligned}$$

where the last inequality is as in the case $r \notin T$.

The second case is if $Pr[\bar{R}] > 1/n$. In this case we **only** sum the contribution of $H = \bar{R}$ and of $r \in H$ (disjoint events), is taken into account.

$$
\begin{aligned}
E[\Delta(R)] &\geq Pr[\bar{R}]\Delta(\bar{R}) + Pr[r \text{ is hit }] \cdot \Delta(\{r\}) \qquad \text{(Corollary 19)} \\
&\geq Pr[\bar{R}]\left(f(\bar{R}) - c_1\right) + Pr[r \text{ is hit }] \cdot \Delta(\{r\}) \qquad \left(\text{We assume } Pr[\bar{R}] \geq \frac{1}{n}\right) \\
&\geq \frac{1}{n} \cdot \left(f(\bar{R}) - c_1\right) + Pr[r \text{ is hit }] \cdot \Delta(\{r\}) \qquad \text{(Lemma 9)} \\
&\geq \frac{1}{n} \cdot \left(f(\bar{R}) - c_1\right) + \Delta(\{r\})/n \qquad \text{(Definition of } \Delta) \\
&= \frac{1}{n} \cdot f(\bar{R}) \qquad \text{(Definition of } f) \\
&\geq \frac{1}{n} p(R).
\end{aligned}
$$

This finishes the proof of Lemma 12 and thus Theorem 3.

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4 Proof of Lemma 16

5 Ratio 5/4 for Min-Power Edge-Cover with unit costs (Theorem 4)

Let $E(T)$ denote the set of edges in E that have both endnodes in T. We say that a star S in G is a **proper star** if all the leaves of S are terminals. Let T_S denote the set of terminals in S. The algorithm for unit costs is as follows.

Algorithm 3: Unit-Costs-Algorithm($G = (V, E), T$) (ratio 5/4)

1 $F \leftarrow E(T)$, $E \leftarrow E \setminus E(T)$, remove from T all terminals covered by $E(T)$
2 while there is a proper star S in G with $|T_S| \geq 4$ do
   $F \leftarrow F \cup S$, $T \leftarrow T \setminus T_S$, $G \leftarrow G \setminus T_S$
3 compute a solution with the KN-Algorithm (with input the current G and T) and add this solution to F
4 return F

In the case of unit costs, if F is a feasible solution then $p(F \cup E(T)) = p(F)$, since $p_F(v) = 1$ for all $v \in T$. This implies that there exists an optimal solution F such that $E(T) \subseteq F$, and thus step 1 in the algorithm is optimal. Note that after this step T is an independent set in G. We may also assume that $V \setminus T$ is an independent set, as edges in $E(V \setminus T)$ do not cover any terminal.

Consider an iteration at step 2 when a proper star S with $k \geq 4$ terminals is chosen. Adding S to F increases $p(F)$ by $k+1$ and removing T_S from G reduces the optimum by at least k. Hence it is a $\frac{k+1}{k} \leq 5/4$ local ratio step. We now show that the KN-Algorithm achieves ratio 5/4 for the residual instance.

Lemma 20 *In the case of unit costs, if T is an independent set in G and if G has no star with 4 terminals then the KN-Algorithm has ratio 5/4.*

Proof. In [18] the following is proved.
If for any proper star S in G there exists a 2-restricted T-cover $\mathcal{T}$ of T_S such that $p(\mathcal{T}) \leq \alpha p(S)$, then the KN-Algorithm achieves ratio α.
In the case considered in the lemma we have unit costs, the center of S is not a terminal, and S has at most 3 leaves. Let u be the center of S and $\{v_1, \ldots, v_q\}$ the set of leaves of S, where $u \notin T$ and $q \in \{1, 2, 3\}$. Note that $p(S) = q + 1$. If $q = 3$ then we take $\mathcal{T} = \{\{v_1\}, \{v_2, v_3\}\}$; then $p(S) = 4$ and $p(\mathcal{T}) = 2 + 3 = 5$. If $q \in \{1, 2\}$ then we take $\mathcal{T} = \{T(S)\}$ and get $p(S) = q + 1 = p(\mathcal{T})$. In both cases $p(\mathcal{T}) \leq \frac{5}{4} p(S)$, and the lemma follows. ■

6 Appendix: Ratio 1.5 for Min-Power Terminal Backup (Theorem 5)

We reduce Min-Power Terminal Backup to the Min-Cost Edge-Cover problem, that is solvable in polynomial time, c.f., [29]. However, the reduction is not approximation ratio preserving, but incurs a loss of 3/2 in the approximation ratio. That is, given an instance (G, c, T) of Min-Power Terminal Backup, we construct in polynomial time an instance (G', c', T) of Min-Cost Edge-Cover such that:

(i) For any T-cover F' in G' corresponds a T-backup F in G with $p(F) \leq c'(F')$.

(ii) $\mathsf{opt}' \leq 3\mathsf{opt}/2$, where opt' is the minimum cost of a T-cover in (G', c', T).

Hence if F' is an optimal (min-cost) solution to (G', c', T), then

$$p(F) \leq c'(F') = \mathsf{opt}' \leq 3\mathsf{opt}/2 \ .$$

Definition 21 *A* **spider** *is a rooted tree such that only its root, called the* **center***, may have degree 3 or more (equivalently, a spider is a subdivision of a star). Given a set T of terminals, we say that a spider S is T-***proper** *if the set $S \cap T$ of its terminals is the set of leaves of S.*

Given an instance of Min-Power Terminal Backup or Min-Cost Terminal Backup, we may assume that all the terminals have degree 1, namely, each $t \in T$ has a unique edge in G incident to it; this is achieved by a standard reduction of adding for every $t \in T$ a new node t' and an edge tt' of cost 0, and making t' a terminal instead of t (note that this reduction does not work for Min-Power Edge-Cover). Under this assumption, we have the following.

Proposition 22 *Let F be an inclusion minimal T-backup. Then any connected component C of the graph $H = (V, F)$ is a T-proper spider.*

Proof. Clearly, F is a tree and every leaf of C is a terminal. If this tree has two vertices of degree at least three, than removing an edge on the path between these two vertices results in a valid T-backup. ■

We now define a certain decomposition of spiders, similar to the decompositions of stars in [18].

Definition 23 *Let S be a spider. A collection $\mathcal{D}$ of paths between the leaves of S such that every leaf belongs to some path is called a* **2-decomposition** *of S. The power $p(\mathcal{D}) = \sum_{S_j \in \mathcal{D}} p(S_j)$ of $\mathcal{D}$ is the sum of the powers of its paths.*

Lemma 24 *Any spider S admits a 2-decomposition $\mathcal{D}$ with $p(\mathcal{D}) \le \frac{3}{2}p(S)$.*

Proof. If S is a path then the statement is obvious, so assume that S has at least 3 leaves. Let s be the center of S, let T' be the set of leaves of S, and let $d = |T'|$ be the number of leaves of S. For each terminal $t_i \in T'$, let s_i be the neighbor of s on the $t_i s$-path (possibly $s_i = t_i$), let $\hat{c}_i$ be the sum of the costs of the edges on the $t_i s$-path, and let $c_i = c(s_i s)$, $i = 1, \ldots, d$. Assume w.l.o.g. that the leaves in T are ordered such that $\hat{c}_1 \le \hat{c}_2 \le \cdots \le \hat{c}_d$. Note that $\hat{c}_i \ge c_i$, and thus $\hat{c}_i \ge \hat{c}_j \ge c_j$ for any $i \ge j$, and that the power of the spider is

$$p(S) = \sum_{i=1}^{d} \hat{c}_i + \max_{1 \le i \le d} c_i$$

We now define our 2-decomposition $\mathcal{D}$. In the case of d even we just take consecutive disjoint pairs $T_i = \{t_{2i-1}, t_{2i}\}$, $i = 1, \ldots, \lfloor d/2 \rfloor$. In the case of d odd, we take the two pairs $\{t_1, t_2\}, \{t_1, t_3\}$ and then add to them the remaining $\lfloor d/2 \rfloor - 1$ consecutive disjoint pairs of the (possibly empty) sequence formed by the remaining terminals in $T \setminus \{t_1, t_2, t_3\}$. Recall that the power of this decomposition is defined to be the sum of the power of its paths, or in other words:

$$p(\mathcal{D}) = \sum_{i=1}^{d} \hat{c}_i + \sum_{i=1}^{d/2} \max\{c_{2i-1}, c_{2i}\} \qquad \text{if } d \text{ is even}$$

$$p(\mathcal{D}) = \sum_{i=1}^{d} \hat{c}_i + \sum_{i=2}^{\lfloor d/2 \rfloor} \max\{c_{2i}, c_{2i+1}\} + \hat{c}_1 + \max\{c_1, c_2\} + \max\{c_1, c_3\} \quad \text{if } d \text{ is odd}$$

We need to prove that $3p(S) \ge 2p(\mathcal{D})$.

If d is even then we need to prove that:

$$3\left(\sum_{i=1}^{d} \hat{c}_i + \max_{1 \le i \le d} c_i\right) \ge 2\left(\sum_{i=1}^{d} \hat{c}_i + \sum_{i=1}^{d/2} \max\{c_{2i-1}, c_{2i}\}\right)$$

By rearranging terms we obtain:

$$\sum_{i=1}^{d} \hat{c}_i + 3 \max_{1 \le i \le d} c_i \ge 2 \sum_{i=1}^{d/2} \max\{c_{2i-1}, c_{2i}\}$$

The latter inequality holds since:

$$\begin{aligned}
\sum_{i=1}^{d} \hat{c}_i + 3 \max_{1 \le i \le d} c_i \quad &\ge \quad \sum_{i=1}^{d/2-1} (\hat{c}_{2i} + \hat{c}_{2i+1}) + 2 \max_{1 \le i \le d} c_i \ge \\
&\ge \quad 2 \sum_{i=1}^{d/2-1} \max\{c_{2i-1}, c_{2i}\} + 2 \max\{c_{d-1}, c_d\} \\
&= \quad 2 \sum_{i=1}^{d/2} \max\{c_{2i-1}, c_{2i}\}
\end{aligned}$$

For the first inequality we applied a standard manipulation of indices, giving up some terms while recalling that all costs and powers are non-negative; the second inequality is since $\hat{c}_{2i+1} \geq \hat{c}_{2i} \geq \max\{c_{2i-1}, c_{2i}\}$; the last equality is obvious.

If d is odd then we need to prove that:

$$3\left(\sum_{i=1}^{d}\hat{c}_i + \max_{1\leq i\leq d} c_i\right) \geq 2\left(\sum_{i=1}^{d}\hat{c}_i + \sum_{i=2}^{\lfloor d/2\rfloor}\max\{c_{2i}, c_{2i+1}\} + \hat{c}_1 + \max\{c_1,c_2\} + \max\{c_1,c_3\}\right)$$

By rearranging terms we obtain that we need:

$$\sum_{i=1}^{d}\hat{c}_i + 3\max_{1\leq i\leq d} c_i \geq 2\hat{c}_1 + 2\max\{c_1,c_2\} + 2\max\{c_1,c_3\} + 2\sum_{i=2}^{\lfloor d/2\rfloor}\max\{c_{2i}, c_{2i+1}\}.$$

If $d > 3$, the latter inequality holds since:

$$\begin{aligned}
\sum_{i=1}^{d}\hat{c}_i + 3\max_{1\leq i\leq d} c_i &= \hat{c}_1 + (\hat{c}_2+\hat{c}_3) + \hat{c}_4 + \hat{c}_d + 3\max_{1\leq i\leq d} c_i + \sum_{i=3}^{\lfloor d/2\rfloor}(\hat{c}_{2i-1}+\hat{c}_{2i}) \\
&\geq (\hat{c}_1+\hat{c}_4) + \hat{c}_d + 2\max\{c_1,c_2\} + 3\max_{1\leq i\leq d} c_i + 2\sum_{i=2}^{\lfloor d/2\rfloor-1}\max\{c_{2i}, c_{2i+1}\} \\
&\geq 2\hat{c}_1 + \max\{c_{d-1},c_d\} + 2\max\{c_1,c_2\} + \\
&\quad \max\{c_{d-1},c_d\} + 2\max\{c_1,c_3\} + 2\sum_{i=2}^{\lfloor d/2\rfloor-1}\max\{c_{2i}, c_{2i+1}\} \\
&= 2\hat{c}_1 + 2\max\{c_1,c_2\} + 2\max\{c_1,c_3\} + 2\sum_{i=2}^{\lfloor d/2\rfloor}\max\{c_{2i}, c_{2i+1}\}.
\end{aligned}$$

The first equality follows by applying a standard manipulation of indices; the first inequality is since $\hat{c}_{i+1} \geq \hat{c}_i \geq \max\{c_{i-1}, c_i\}$; the last inequality is since $\hat{c}_4 \geq \hat{c}_1$, $\hat{c}_d \geq \max_{1\leq i\leq d} c_i \geq \max\{c_i, c_j\}$ for any i, j, and the last equality follows by applying a standard manipulation of indices.

If $d = 3$, we must prove that:

$$\hat{c}_1 + \hat{c}_2 + \hat{c}_3 + 3\max_{1\leq i\leq 3} c_i \geq 2\hat{c}_1 + 2\max\{c_1,c_2\} + 2\max\{c_1,c_3\},$$

and this follows since $\hat{c}_2 \geq \hat{c}_1$ and $\hat{c}_3 \geq \max\{\hat{c}_1, \hat{c}_3\} \geq 2\max\{c_1, c_3\}$. This completes the proof of the lemma. ■

Our algorithm for Min-Power Terminal Backup is as follows.

Algorithm 4: Approx-T-Backup$(G=(V,E), c, T)$ (ratio 3/2)

1. **construct** an instance $(G'=(T,E'), c')$ of Min-Cost T-Cover as follows.
   - For every $\{t_i, t_j\} \in T$ with $i \neq j$ let L_{ij} be a t_it_j-path of minimum power.
   - The graph G' is a complete graph on T with edge costs $c'(t_it_j) = p(L_{ij})$.
2. **compute** a minimum cost T-cover F' in G', c'.
3. **return** $F = \cup\{L_{ij} : t_it_j \in F'\}$

We note that the problem of computing a minimum power t_it_j-path can be solved in polynomial time by a simple reduction to its min-cost variant, c.f. [1, 20]. All the other parts of the algorithm can also be implemented in polynomial time. The following statement is used to prove that the approximation ratio of the algorithm is 3/2.

Lemma 25

(i) *If F' is a T-cover in G' then $F = \cup\{L_{ij} : t_it_j \in F'\}$ is a T-backup in G and $p(F) \leq c'(F')$.*

(ii) $\mathsf{opt}' \leq 3\mathsf{opt}/2$*, where* opt' *is the minimum cost of a T-cover in G', c'.*

Proof. F is a T-backup since F' is a T-cover, and since L_{ij} connects t_i and t_j for every $t_it_j \in F'$. Also, $p(F) \leq c'(F')$ since

$$p(F) = p\left(\bigcup_{t_it_j \in F'} L_{ij}\right) \leq \sum_{t_it_j \in F'} p(L_{ij}) = \sum_{t_it_j \in F'} c'(t_it_j) = c(F') \ .$$

We now prove that $\mathsf{opt}' \leq 3\mathsf{opt}/2$. Let F be an optimal inclusion minimal solution to Min-Power Terminal Backup in (G, c, T), so $p(F) = \mathsf{opt}$. By Lemma 24 there exists a 2-decomposition $\mathcal{D}$ of F with $p(\mathcal{D}) \leq 3p(F)/2 = 3\mathsf{opt}/2$. To every path $L_{ij} \in \mathcal{D}$ corresponds an edge $e_{ij} = t_it_j$ in G' and $c'(e_{ij}) \leq p(L_{ij})$. Let $F' = \{e_{ij} : L_{ij} \in \mathcal{D}\}$. Then F' is a T-cover in G', since e_{ij} and L_{ij} have the same endnodes t_i, t_j, and since F is a T-cover. Hence $\mathsf{opt}' \leq c'(F')$. Thus:

$$\mathsf{opt}' \leq c'(F') = \sum_{e' \in F'} c'(e') \leq \sum_{i,j} p(L_{ij}) = p(\mathcal{D}) \leq 3p(F)/2 = 3\mathsf{opt}/2 \ .$$

■

Theorem 5 now easily follows from Lemma 25. Let F, F' be as in the algorithm. Then, by Lemma 25, we have $p(F) \leq c'(F') = \mathsf{opt}' \leq 3\mathsf{opt}/2$.

The proof of Theorem 5 is complete.

7 Appendix: Removing the need of k-restricted approach

We can obtain T-covers as in Theorem 3 without using k-restricted covers. This improves the running time of the main approximation algorithm.

The "hypergraphic" linear program $LP(T)$ below has a variable x_R for every star R and it is a relaxation for the problem of finding a T-cover of minimum power. Let $\mathcal{R}$ be the collection of all stars of the input graph.

$$\begin{array}{lll} \min & \displaystyle\sum_{R \in \mathcal{R}} p(R) x_R & \\ \text{s.t.} & \displaystyle\sum_{R \in \mathcal{R}, v \in V(R)} x_R \geq 1 & \forall v \in T \\ & x_R \geq 0 & \forall R \in \mathcal{R} \end{array}$$

The approximation algorithm is the same as in Theorem 3, but it uses $LP(T)$ instead of $LP_k(T)$. The remaining challenge is solving $LP(T)$, which has has exponentially many variables.

For this, we use an auxiliary linear program $ALP(T)$, which has variables $y_{(v,u)}$ for all 2-tuples (v,u) with $v, u \in V$ with $v \neq u$ and $uv \in E(G)$, and $z_{(w,v,u)}$ for all 3-tuples (w,v,u)) with $w, v, u \in V$, $u \notin \{v, w\}$ and $c(wu) \leq c(vu)$ (note that $w = v$ is possible). $ALP(T)$ is:

$$
\begin{array}{lll}
\min & \sum_{(v,u)} c(uv) y_{(v.u)} + \sum_{(w,v,u)} c(wu) z_{(w,v,u)} & \\
\text{s.t.} & \sum_{v} y_{(v.u)} + \sum_{(w,v) \mid c(uv) \leq c(wv)} z_{(u,w,v)} \geq 1 & \forall u \in T \\
& z_{(w,v,u)} \leq y_{(v,u)} & \forall (w,v,u) \\
& z_{(v,v,u)} = y_{(v,u)} & \forall (v,u), v \neq u \\
& y_{(v,u)} \geq 0 & \forall (v,u), v \neq u \\
& z_{(w,v,u)} \geq 0 & \forall (w,v,u), v \neq u \text{ and } c(wu) \leq c(vu)
\end{array}
$$

Let us check this equivalence. If we have a solution to $LP(T)$, from variables x_R we obtain the variables $y_{(v,u)}$ and $z_{(w,v,u)}$ in $ALP(T)$ as follows: we start with each such variable as 0, and for every R, star with center u and leaves v_1, v_2, ..., v_k arranged in non-increasing order of costs $c(uv_1) \geq c(uv_2) \geq \cdots \geq c(uv_k)$, we add x_R to $y_{(v_1,u)}$ and to $z_{(v_j,v_1,u)}$, for $j = 1$ to k. Notice that $p(R) = c(uv_1) + \sum_{j=1}^{k} c(uv_j)$, which is at most the increase in the objective function of $ALP(T)$. Also, notice that $z_{(v,v,u)} = y_{(v,u)}$ for all $v \neq u$ since every time $y_{(v,u)}$ is increased, $z_{(v,v,u)}$ is increased by the same amount. Moreover, the constraint, for a given $u \in T$, $\sum_v y_{(v.u)} + \sum_{(w,v)} z_{(u,w,v)} \geq 1$ is satisfied, as for every star R with $u \in V(R)$, x_R contributes to either $y_{(v_1,u)}$ (when u is the center of R), or $z_{(u,w,v)}$ (when v the center of R and w the first child of R), and using $\sum_{R \in \mathcal{R}, v \in V(R)} x_R \geq 1$.

Now suppose we have a feasible solution to $ALP(T)$. One by one, go through all 2-tuples (v,u) with $v \neq u$. Let $w_1, w_2, \ldots, w_q$ be the vertices with $c(w_j u) \leq c(vu)$ sorted such in non-decreasing order of $z_{(w_i,v,u)}$. As for all i, we have that $z_{(w_i,v,u)} \leq y_{(v,u)} = z_{(v,v,u)}$, we may assume that $v = w_q$. For $i = 1, 2, \ldots, q$, star $R_i = R_i(v,u)$ will have center u and children $w_i, \ldots, w_q$.Set $x_{R_1} = z_{(w_1,v,u)}$, and for $i > 1$ set $x_{R_i} = z_{(w_i,v,u)} - z_{(w_{i-1},v,u)}$. Using that $c(w_j u) \leq c(vu)$ for all u, we have that $p(R_i) \leq c(vu) + \sum_{j=i}^{q} c(w_i u)$. Using that $y_{(v,u)} = z_{(w_q,v,u)}$, we deduce that:

$$
\begin{aligned}
\sum_{i=1}^{q} p(R_i) x_{R_i} &\leq z_{(w_1,v,u)} \left(c(vu) + \sum_{j=1}^{q} c(w_i u) \right) + \\
& \qquad \sum_{i=2}^{q} \left(z_{(w_i,v,u)} - z_{(w_{i-1},v,u)} \right) \left(c(vu) + \sum_{j=i}^{q} c(w_i u) \right) \\
&= c(vu) z_{(w_q,v,u)} + \sum_{i=1}^{q} z_{(w_i,v,u)} c(w_i u) \\
&= c(vu) y_{(v,u)} + \sum_{i=1}^{q} z_{(w_i,v,u)} c(w_i u).
\end{aligned}
$$

Therefore, while doing this for all 2-tuples (v,u) with $v \neq u$, we obtain a solution of $LP(T)$ without increasing the costs.

Moreover, when creating these stars R_i, we have that

$$\sum_{i,u\in V(R_i)} x_{R_i} = y_{(v,u)}$$

and

$$\sum_{i,w_j\in V(R_i)} x_{R_i} = z_{(w_j,v,u)}.$$

Therefore, for any u, we have $\sum_{R\in\mathcal{R},u\in V(R)} x_R \geq \sum_v y_{(v,u)} + \sum_{(w,v)} z_{(u,w,v)} \geq 1$, and thus our $LP(T)$ solution is feasible.

Thus, with $LP(T)$ we eliminated the need for k-restricted covers; by comparisons for the Min-Cost Steiner Tree problem, so far, k-restricted decompositions are still needed for the best ratios, or for any linear program with ratio provable better than 2.

8 Conclusions

It is quite possible that solving one $LP(T)$ is enough (when applying iterative rounding, avoid resolving new linear programs), as it was shown for Min-Cost Steiner Tree by [10].

We also believe that $LP(T)$ has integrality ratio better than 3/2, but with the methods of this paper we were not able to obtain the ratio of Theorem 3, the general case, with respect to $LP(T)$ ($LP(T)$ may be used to bound the output of the iterative rounding algorithm, but the KN-Algorithm's output is directly compared to the optimum) .

One may be able to improve the running time of Lemma 6 to $2^{O(t)}n^{O(1)}$ using color-coding and dynamic programming. This is not critical for this paper.