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A high order ADI method for separable generalized Helmholtz equations

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Abstract

We present a multilevel high order ADI method for separable generalized Helmholtz equations. The discretization method we use is a one-dimensional fourth order compact finite difference applied to each directional component of the Laplace operator, resulting in a discrete system efficiently solvable by ADI methods. We apply this high order difference scheme to all levels of grids, and then starting from the coarsest grid, solve the discretized equation with an ADI method at each grid level, with the solution from the previous grid level as the initial guess. The multilevel procedure stops as the ADI finishes its iterations on the finest grid. Analytical and experimental results show that the proposed method is highly accurate and efficient while remaining as algorithmically and data-structurally simple as the single grid ADI method. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Generalized Helmholtz equation; ADI method; Multilevel method; High order discretization

1. Introduction

In this paper, we consider the numerical solution of separable generalized Helmholtz equations

$$\begin{cases} -\Delta u + f(x)u + g(y)u = r(x,y), & (x,y) \in \Omega, \\ u(x,y) = b(x,y), & (x,y) \in \partial\Omega, \end{cases} \tag{1}$$

on a rectangular domain Ω for $f(x) \geq 0$ and $g(y) \geq 0$. Such problems arise in the linearization process or the Picard process of semilinear elliptic equations like the steady state reaction–diffusion equations.

Eq. (1) can be solved using the following Picard iterative process employed in Ref. [4]

$$(-\Delta + \alpha)u^n = [\alpha - f(x) - g(y)]u^{n-1} + r, \tag{2}$$

where at each iteration Eq. (2) is a Helmholtz equation solvable by FFT based direct methods [2,6,11]. This Picard process allows high order discretization and has a good computational complexity of $O(h^{-2}(\log h)^2)$ on grids with mesh size h.

In this paper, we present a multilevel high-order alternating direction implicit (ADI) solution method that has a complexity of $O(-h^{-2}(\log h))$. We first discretize the equation by applying a one-dimensional (1-D) fourth order finite difference to each directional component of the Laplace operator. Unlike the popular 9-point square stencil scheme or the 9-point cross stencil fourth order finite central difference scheme, our discretization can be easily combined with ADI methods to produce an accurate and efficient solver. Further more, the directional discretization technique is applicable to three-dimensional space (3-D) as easily as to the two-dimensional space (2-D), without any derivation or coefficient calculation effort. Starting from the grid with the coarsest meshes, we then apply the high-order difference scheme to all levels of grids and solve the discretized equation using ADI at each grid level, with the solution from the previous grid level as the initial guess. So our multilevel method is simple and easily implementable—it requires simple data structure and incurs almost no data structure handling overhead. ADI method has a complexity of $O(h^{-2}(\log h)^2)$ for a given error tolerance compatible with the discretization error. Our analysis shows that the multilevel treatment reduces the complexity by a factor of $O(-\log h)$ to $O(-h^{-2}\log h)$.

Our discussion of the newly proposed solution method is focused on the 2-D problem. However as briefly discussed in the paper, both the discretization technique and the multilevel ADI are extendible to 3-D space with the same computational efficiency and data structure simplicity.

This paper is organized as follows. Section 2 presents a fourth order directionally independent discretization of the Laplace operator that allows efficient ADI methods. The single grid and multilevel high order ADI methods and their complexity analyses are discussed in Section 3. Testing results are presented and discussed in Section 4 and Section 5 gives the conclusion.

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2. High order discretization

ADI methods seem difficult to be applicable to discrete systems obtained via high order discretization since the popular fourth order 9-point scheme for the Laplace operator

$$-\begin{pmatrix} & \frac{1}{8} & \\ \frac{1}{8} & 1 & \frac{1}{8} \\ & \frac{1}{8} & \end{pmatrix}\Delta p^{i,j} = h^{-2}\begin{pmatrix} -\frac{1}{4} & -1 & -\frac{1}{4} \\ -1 & 5 & -1 \\ -\frac{1}{4} & -1 & -\frac{1}{4} \end{pmatrix}p^{i,j}$$

has a square stencil, difficult to be combined directly with the ADI method to produce an efficient solver. And the cross stencil fourth order finite central difference

$$-\Delta p^{i,j} = h^{-2}\begin{pmatrix} & & \frac{1}{12} & & \\ & & -\frac{4}{3} & & \\ \frac{1}{12} & -\frac{4}{3} & 5 & -\frac{4}{3} & \frac{1}{12} \\ & & -\frac{4}{3} & & \\ & & \frac{1}{12} & & \end{pmatrix}p^{i,j}$$

not only has a boundary problem [3] but also its penta-diagonal directional components cannot be as efficiently solved as tri-diagonal systems.

However, 1-D compact finite difference schemes [5,7] for second order derivatives have been available for a long time. One of them is the following fourth order scheme

$$\frac{1}{12}u''_{i-1} + \frac{5}{6}u''_i + \frac{1}{12}u''_{i-1} = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}. \quad (3)$$

Applying this finite difference to Eq. (1) with the boundary condition incorporated, we obtain

$$(-L_{m-1} + F) \otimes I_{n-1}U + I_{m-1} \otimes (-L_{n-1} + G)U = R, \quad (4)$$

where L_n is an $n \times n$ matrix given by

$$L_n = \frac{1}{h^2}\begin{pmatrix} \frac{5}{6} & \frac{1}{12} & 0 & \cdots & 0 \\ \frac{1}{12} & \frac{5}{6} & \frac{1}{12} & \cdots & 0 \\ . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & \frac{1}{12} \\ 0 & 0 & 0 & \cdots & \frac{5}{6} \end{pmatrix}^{-1} \times \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & -2 \end{pmatrix},$$

I_n denotes the identity matrix in n-dimensional space, $\otimes$ is the tensor product notation [9], F and G are diagonal matrices corresponding to the functions $f(x)$ and $g(y)$, respectively, and U is the solution vector.

Birkhoff and Varga showed in Ref. [1] that when the directional components $(-L_{m-1} + F) \otimes I_{n-1}$ and $I_{m-1} \otimes (-L_{n-1} + G)$ of the discretized equation are symmetric, positive definite and commutative, the Peaceman–Rachford [10] ADI method has a convergence rate of $O(-(\log h)^{-1})$. Thus these positive definite and commutative conditions are essential for a high convergence rate. The commutativity condition is obvious with the tensor product notation. The positive definite condition also holds, as we shall see below.

Remark 1. *The matrices* $(-L_{m-1} + F) \otimes I_{n-1}$ *and* $I_{m-1} \otimes (-L_{n-1} + G)$ *are symmetric and positive definite.*

Proof. We prove the above statement only for the matrix $(-L_{m-1} + F) \otimes I_{n-1}$. The proof for the matrix $I_{m-1} \otimes (-L_{n-1} + G)$ is similar and thus omitted.

Since F is diagonal and positive definite, it suffices to show that $-L_{m-1}$ is positive definite. Now let P_{m-1} denote the matrix

$$P_{m-1} = \begin{pmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,m-2} & p_{1,m-1} \\ p_{2,1} & p_{2,2} & \cdots & p_{2,m-2} & p_{2,m-1} \\ p_{3,1} & p_{3,2} & \cdots & p_{3,m-2} & p_{3,m-1} \\ . & . & \cdots & . & . \\ p_{m-1,1} & p_{m-1,2} & \cdots & p_{m-1,m-2} & p_{m-1,m-1} \end{pmatrix}$$

where $p_{i,j} = \sin((ij\pi)/m)$ for $i,j = 1,2,\cdots,m-1$. A straightforward calculation shows that $P_{m-1}^{-1}L_{m-1}P_{m-1} = \Lambda$, where Λ is a diagonal matrix given by

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 & 0 \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & \lambda_{m-2} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_{m-1} \end{pmatrix},$$

and

$$\lambda_i = \frac{12\cos\left(\frac{i\pi}{m}\right) - 12}{5 + \cos\left(\frac{i\pi}{m}\right)}h^{-2} < 0,$$

for $i = 1,2,\cdots,m-1$. For matrix P_{m-1}, we have that $P_{m-1}^{-1} = (2/m)P_{m-1}^T$. Thus, $P_{m-1}^T L_{m-1} P_{m-1} = (m/2)\Lambda$, and $L_{m-1} = (2/m)P_{m-1}\Lambda P_{m-1}^T$. Therefore $-L_{m-1}$ is symmetric and positive definite. □

It is clear from the proof given above that these symmetric and positive definite properties will also hold for 3-D

problems when the compact finite difference (3) is applied to the 3-D Laplace operator.

3. Multilevel high order ADI methods

3.1. Single grid high-order ADI

Applying the Peaceman–Rachford ADI method to the discrete Eq. (4) on a rectangular grid with uniform meshs h_x and h_y on the x- and y-directions, respectively, we obtain

$$\begin{cases} (\rho_n + A_1)U^{n+(1/2)} = (\rho_n - A_2)U^n + R \\ (\rho_n + A_2)U^{n+1} = (\rho_n - A_1)U^{n+(1/2)} + R \end{cases} \tag{5}$$

where A_1 and A_2, respectively, denote the two terms $(-L_{m-1} + F) \otimes I_{n-1}$ and $I_{m-1} \otimes (-L_{n-1} + G)$ on the left hand side of Eq. (4). By the analysis in Section 2, A_1 and A_2 are symmetric, positive definite, and commutative. Therefore, the ADI method (5) with Wachspress parameters [12,13] has a convergence rate of $O(-(\log h)^{-1})$ for $h = \max\{h_x, h_y\}$. Thus, denoting the convergence rate on a grid with mesh size h by $r(h)$, we can assume that

$$r(h) \geq -C_0(\log h)^{-1} \tag{6}$$

for some constant $C_0 > 0$.

If the error of the initial guess is $e_0(h)$ and the iteration stopping error tolerance is $e_s(h)$, then by the definition of the convergence rate [8], the number of iterations equals

$$\frac{\log (e_0(h)/e_s(h))}{r(h)}.$$

With estimate (6), we arrive at

$$\text{Iterations} \leq C_1(\log h)\log \frac{e_s(h)}{e_0(h)}, \tag{7}$$

where $C_1 = 1/C_0$. Each ADI iteration consists of solving four tri-diagonal systems, two for evaluating the right hand sizes of Eq. (5) necessitated by using the compact finite difference (3), and two for inverting $(\rho_n + A_1)$ and $(\rho_n + A_2)$ in the left hand sides of Eq. (5). Solving these four tri-diagonal systems is the major computation cost of each ADI iteration. Hence the computation cost per ADI iteration is $O(h^{-2})$ and bounded by $C_2' h^{-2}$ for some constant $C_2' > 0$. Then by Eq. (7), the computation cost for the ADI method (5) to stop at tolerance $e_s(h)$ with a starting error of $e_0(h)$ is

$$C_2 h^{-2}(\log h)\log \frac{e_s(h)}{e_0(h)}, \tag{8}$$

where $C_2 = C_1 C_2'$. For instance, if the error tolerance is chosen to be $e_s(h) = h^4$ and the initial error is assumed to be $e_0(h) = 1$, then the ADI has a complexity of $O(h^{-2}(\log h)^2)$.

For 3-D problems, we choose the following ADI for discretized equations $(A_1 + A_2 + A_3)U = R$,

$$U^{n+1} = U^n - 2\rho_n^2 Q(\rho_n)\{(A_1 + A_2 + A_3)U^n - R\}, \tag{9}$$

where Q is a matrix function given by

$$Q(\rho) = (A_1 + \rho)^{-1}(A_2 + \rho)^{-1}(A_3 + \rho)^{-1}.$$

When A_1, A_2 and A_3 are symmetric, positive definite and pairwisely commutative, Douglas gave a set of parameters [4] with which the 3-D ADI method (9) has a convergence rate also of $O(-(\log h)^{-1})$. Then its complexity is

$$O\left(h^{-3}(\log h)\log (e_s(h)/e_0(h))\right)$$

by an analysis similar to that for the 2-D ADI.

3.2. Multilevel ADI

With a rectangular mesh applied to the domain, we obtain a grid, which is designated as the finest grid. We choose every other grid point in each direction to form the next coarser level grid, a subset of the original grid with double mesh sizes, and no extra data structure is needed for this grid level. By the same procedure, we designate a sequence of grids, each a subset of the immediate finer level grid with double mesh sizes. All of them need no extra data structure other than an index of one single integer number. The number of grid levels is chosen to be of $O(-\log_2 h)$, namely, the coarsest grid has a size close to 2×2.

Our multilevel procedure starts from the coarsest grid level. On all the grids, applying discretization scheme (3) to Eq. (1), we obtain discrete equations with the same matrix form as Eq. (4) of different problem sizes. ADI method (5) is then employed to solve the discrete equation on each grid level, with the solution of the equation at the immediate coarser grid level interpolated to the current grid level as the initial guess. The interpolation method is chosen in such a way that it has a fourth order interpolation accuracy, and the ADI iteration stopping tolerance e_s is chosen to be of the same order as that of the discretization. This multilevel ADI method solves the equation from the coarsest grid level to the finest, and the whole multilevel process ends after the ADI method finishes its iterations on the finest level.

Unlike conventional multigrid methods, our newly proposed solver is a one-way multilevel method, of both algorithmic and data-structural simplicity. Its implementation needs only one subroutine and one loop more than the single grid ADI method (5)—an interpolation subroutine and a loop that goes through all grid levels. Such simplicity provides great potential for its applicability in complex systems and in combination with parallel and/or domain decomposition methods.

While conventional multilevel methods are mainly utilizing the smoothing effect [14] of the single grid solvers (or relaxation schemes in the language of multilevel methods) to achieve computation reduction, our method relies on the initial error reduction via interpolation from coarser grids to reduce the iteration numbers on finer grids. To analyze the computation complexity of this multilevel ADI method

more closely, we denote the true solution of the original differential Eq. (1) by u, denote the exact numerical solution of the discrete Eq. (4) on a grid with mesh size h by u_h, denote the approximate ADI solution of the discrete equation at the same grid level by U_h, and denote the initial guess solution interpolated from the immediate coarser grid to the current grid by $\hat{U}_h$.

The discretization scheme (3) is of order four, which means that there exists a constant $C_3 > 0$ such that

$$\| u_h - u \| \leq C_3 h^4.$$

The iteration stopping error tolerance $e_s(h)$ is chosen to be of the same order as that of the discretization. Thus, we have that

$$\| U_h - u_h \| \leq C_4 h^4$$

for some constant $C_4 > 0$. Therefore,

$$\| U_h - u \| \leq \| U_h - u_h \| + \| u_h - u \| \leq (C_3 + C_4) h^4.$$

This inequality holds for every grid level. Hence for the immediate coarser grid, we have that

$$\| U_{2h} - u \| \leq 16(C_3 + C_4) h^4.$$

The solution at the immediate coarser grid is interpolated to the current grid as the initial guess solution in such a way that it maintains fourth order accuracy, i.e. this initial guess $\hat{U}_h$ satisfies

$$\| \hat{U}_h - U_{2h} \| \leq C_5 h^4$$

for some constant $C_5 > 0$. By definition, the error of the initial guess is $\| \hat{U}_h - u_h \|$, so

$$\begin{aligned} e_0(h) = \| \hat{U}_h - u_h \| &\leq \| \hat{U}_h - U_{2h} \| + \| U_{2h} - u \| \\ &\quad + \| u - u_h \| \leq C_5 h^4 + 16(C_3 + C_4) h^4 + C^3 h^4 \\ &= (17C_3 + 16C_4 + C_5) h^4. \end{aligned}$$

Then the ratio of the initial error to the iteration stopping error tolerance satisfies

$$e_0(h) : e_s(h) \leq \frac{17C_3 + 16C_4 + C_5}{C_4}. \tag{10}$$

Then by Eq. (7), the number of iterations needed at grid level with mesh size h is

$$\text{Iterations} \leq C_1(\log h) \log \frac{C_4}{17C_3 + 16C_4 + C_5} = -C_6 \log h, \tag{11}$$

where the positive constant

$$C_6 = C_1 \log \frac{17C_3 + 16C_4 + C_5}{C_4}.$$

And by estimates (8) and (10), the computation cost of the ADI iterations at grid level with mesh size h is $-C_7 h^{-2} \log h$ for some $C_7 > 0$. The interpolation cost is proportional to the number of grid points, and thus can be assumed to be $C_8 h^{-2}$. Then the total computation cost on a grid of mesh size h is bounded by

$$(C_8 - C_7 \log h) h^{-2} \leq -C_9 h^{-2} \log h \tag{12}$$

for some constant $C_9 > 0$.

The above estimates of iteration numbers and computation costs on a grid of mesh h are valid for all levels except the coarsest grid level, since on the coarsest level, the initial guess is not obtained from interpolation. Thus the ratio of initial error to error tolerance may not necessarily satisfy expression (10) and hence ADI could possibly take more iterations than expression (11) on the coarsest grid. However, since the number of levels is chosen in such a way that the coarsest grid has very few grid points as described in the first paragraph of this section, the computation cost on the coarsest grid is of O(1) and thus negligible.

Add up the computation costs at all grid levels, we obtain

$$\begin{aligned} &-C_9\{h^{-2} \log h + (2h)^{-2} \log (2h) + (4h)^{-2} \log (4h) + \cdots\} \\ &\leq -C_9 \log h \{h^{-2} + h^{-2}/2^2 + h^{-2}/4^2 + \cdots\} \end{aligned}$$

for the total computation cost, which is bounded by $-1.4C_9 h^{-2} \log h$ by an easy calculation. Thus, we have shown that the computation complexity of this multilevel ADI is $O(-h^{-2} \log h)$.

4. Numerical testing

Two equations with known solutions have been chosen to test the accuracy and efficiency of the High-order ADI (HADI) and the Multilevel High-order ADI methods (MHADI). The two problems are

(i) $-\Delta u + (e^{\cos(x)} + \cos(y) + y)u = r$
with $u(x, y) = (x + y)^{3.5}(\cos(x) - 1)$;

(ii) $-\Delta u + (\cos(x) + e^y)u = r$
with $u(x, y) = \sin(2x + 3y)$.

Experimental tests have been conducted on an IBM RS/6000 to measure the number of iterations, execution time, and numerical errors. For comparison, we have also solved the two problems using the Picard method (2) with a fourth order FFT solver at each Picard iteration.

In the tests, the coarsest of the MHADI method is chosen to be of size 2×2, and the iteration stopping criterion for all three methods (HADI, MHADI and Picard) is chosen to be the difference between the approximate solution at two consecutive iterations, which is set to h^4 for the first problem and set to $0.1h^4$ for the second problem. We choose different error tolerance because the solution of the first problem is less smooth than the second and thus have a larger discretization error. The testing is conducted on the square domain $[0, 8] \times [0, 8]$ with the same uniform mesh size h on each dimension. $N = 8/h$ is the number of grid points on each x- and y-dimensions.

Table 1
$-\Delta u + (e^{\cos(x)} + \cos(y) + y)u = r$ with $u = (x + y)^{3.5}(\cos(x) - 1)$

Method	Grid Size	16	32	64	128	256	512
MHADI	Iterations[a]	5	7(6)	8(8)	7(9)	8(10)	9(11)
	Time (s)	0.01	0.06	0.31	1.19	5.66	29.3
	Error	3.1×10^{-1}	2.0×10^{-2}	1.3×10^{-3}	1.2×10^{-4}	6.8×10^{-6}	4.0×10^{-7}
	Order		3.9	3.9	3.4	4.1	4.1
HADI	Iterations	9	16	21	26	35	42
	Time (s)	0.02	0.13	0.66	3.40	20.8	110
	Error	3.2×10^{-1}	2.0×10^{-2}	1.3×10^{-3}	8.2×10^{-5}	5.0×10^{-6}	3.2×10^{-7}
	Order		4.0	3.9	4.0	4.0	4.0
Picard	Iterations	17	21	26	30	35	39
	Time (s)	0.03	0.12	0.55	2.68	12.7	66.0
	Error	3.3×10^{-1}	2.1×10^{-2}	1.3×10^{-3}	8.3×10^{-5}	5.1×10^{-6}	3.2×10^{-7}
	Order		4.0	4.0	3.9	4.0	4.0

[a] The number in parenthesis is the predicted iteration numbers.

If a numerical method is of order s, then the error will decrease at a rate of $(1/2)^s$ when a uniformly spaced grid doubles its grid points. We use this information to define our metric

$$\text{Order}(n, 2n) = \log_2 \frac{\text{Error}(n)}{\text{Error}(2n)}$$

to measure the order of accuracy of our numerical solutions. The *log* plot of error against grid size (or mesh size) is usually used to measure the order of a numerical method. The metric Order used in Tables 1 and 2 gives the value of the slope of the *log* plot of the error vs. grid size between each two neighbouring testing grid sizes. Since the slope of a curve is difficult to be exactly visually determined, the metric order is a clearer quantitative indication of the order of a numerical method.

Tables 1–2 present the experimental results for the two testing problems. For MHADI method, the *Iteration* row lists the ADI iterations on each grid level. The number in parenthesis is calculated using formula (11) with the unknown constant C_6 calculated using the iteration number at $N = 16$. For the HADI method and the Picard method, the *Iteration* rows list the total ADI iterations and total Picard iterations, respectively. For all three methods, the *Time* rows give the CPU time (in seconds) taken to solve the problems, and the *Error* rows show the maximal errors of the numerical solutions for the two problems solved on the grid of the indicated size. The experimental results show that the newly proposed multilevel high order ADI method is both accurate and efficient, matching fairly well with our formal analysis.

The Picard process fails to reach the prescribed error tolerance within 1000 iterations for the second problem, because the values of $f(x) + g(y)$ (which is equal to $\cos(x) + e^y$ for problem 2) cover a very large range on square $[0, 8] \times [0, 8]$. The Picard process (2) converges slowly when $f(x) + g(y)$ has a large range of values. For the first problem, $f(x) + g(y) = e^{\cos(x)} + \cos(y) + y$, having a much smaller range of values and the Picard method converges to adequate accuracy within a reasonable number of iterations for each grid size tested.

Fig. 1 is the plot of iterations versus grid size. The grid size is in log scale. The plot shows that for the MHADI

Table 2
$-\Delta u + (\cos(x) + e^y)u = r$ with $u = \sin(2x + 3y)$

Method	Grid Size	16	32	64	128	256	512
MHADI	Iterations	5	6(6)	6(8)	6(9)	7(10)	8(11)
	Time (s)	0.01	0.06	0.25	0.99	4.94	25.2
	Error	1.7×10^{-2}	1.3×10^{-3}	7.1×10^{-5}	5.1×10^{-6}	2.9×10^{-7}	2.1×10^{-8}
	Order		3.7	4.2	3.8	4.1	3.8
HADI	Iterations	6	12	14	20	24	32
	Time (s)	0.01	0.09	0.43	2.58	13.0	81.6
	Error	2.5×10^{-2}	1.1×10^{-3}	3.2×10^{-4}	4.7×10^{-6}	5.2×10^{-7}	1.7×10^{-8}
	Order		4.5	1.8	6.1	3.2	4.9
Picard	Iterations	371	316	742	1000[a]	–	–
	Time (s)	0.57	1.73	16.1	89.0	–	–
	Error	1.3×10^{-2}	4.7×10^{-2}	4.2×10^{-3}	1.4×10^{-3}	–	–

[a] The Picard method fails to reach the error tolerance within 1000 iterations.

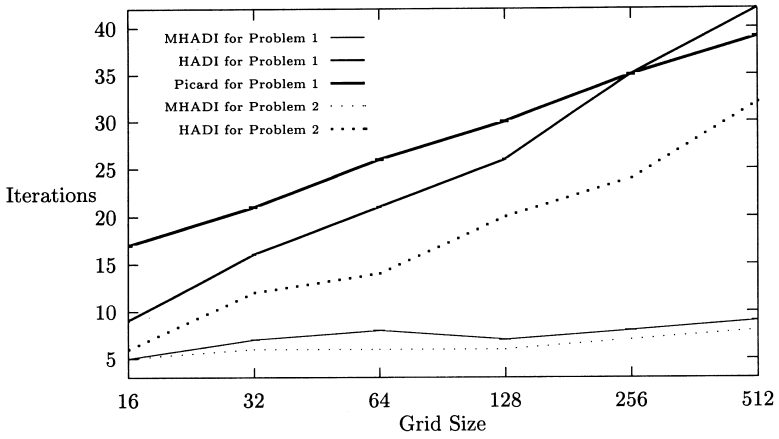

Fig. 1. Iterations vs. grid size.

method the iteration numbers on the finest grid increase slowly, slower than those of both the HADI and the Picard methods. Such slow increase of iteration numbers is also evident from the tables. From grid size 16–512, both the HADI and the Picard methods have more than doubled their iteration numbers while iterations of the MHADI method is less than doubled. In addition to the calculated values of the metric Order in the two tables, we also provide the *log* plot of the error versus the grid size in Fig. 2. The slopes of the curves indicate the orders of the numerical methods.

5. Conclusion

In this paper, we have proposed a one-way multilevel high order ADI method for separable generalized Helmholtz equations. The equation is first discretized by a fourth order 1-D compact finite difference scheme, resulting in a linear system efficiently solvable by ADI methods. Then we designate a sequence of grids, each a subset of its immediate finer grid with double mesh sizes. Starting from the coarsest grid, the fourth order discretization is applied to the differential equation and ADI is employed to solve the discrete system on each grid level, with the solution from the previous level interpolated as the initial guess. The multilevel procedure stops as the ADI finishes its iterations on the finest grid. This proposed method is shown to be highly accurate and efficient by formal analysis and numerical experiments.

The design of this multilevel method is focused on maintaining the algorithmic and data-structural simplicity. Thus the multilevel treatment is carried out in a coarse-to-fine manner to reduce the initial errors on all grids via interpolation. However, we believe that the single grid ADI solver can also be used to design an effective smoother to further reduce the computational complexity by combining a one way coarse-to-fine multilevel cycle for the solution with a V-cycle for residuals, and we leave this investigation as a future study.

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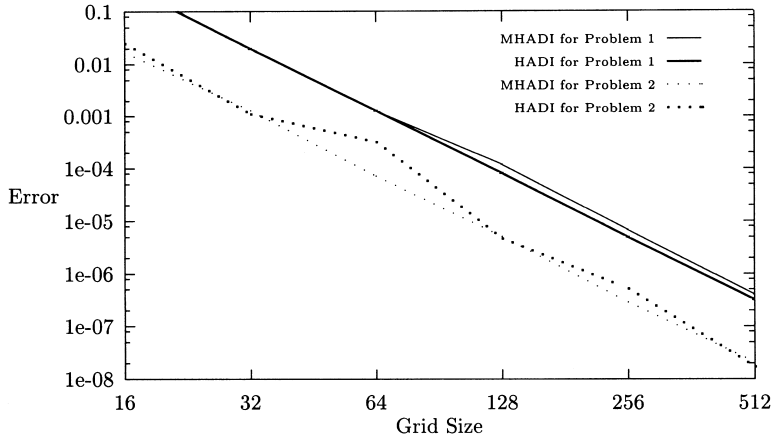

Fig. 2. Error vs. grid size.

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