Fault Tolerant Sensor Networks with Bernoulli Nodes

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***Abstract*— Connectivity, power consuming and fault tolerance are three critical issues in sensor networks. In this paper, sensor networks are modeled by the unit disc graph, random point process and Bernoulli nodes. A sensor network is composed of n identical nodes randomly located on an unit area disc with uniform distribution. Each node is associated with probability p ($0 < p \leq 1$) to be active and with probability $q = 1 - p$ to be inactive. r_n is the transmission radius. As n goes to infinite and let $\pi {r_n}^2 = \frac{\ln n + \xi}{pn}$, then the probability that each node neighbors at least one active node is $e^{-(e^{-\xi})}$, the Gumbel extreme-value function, for $\forall \xi \in R$, and the cardinality of one connected component in the random graph $G(n, r_n, p)$ is either 1 or tending to infinite.**

***Index Terms*— sensor network, ad hoc network, geometry random graph, random point process, unit disc graph, Bernoulli node, connectivity, fault tolerance**

I. Introduction

A sensor network is composed of many similar sensor nodes with limited resources. Two nodes have a direct link if and only if both are within the other's transmission range. In order to communicate with other nodes outside of the transmission range, one node needs to rely on its neighbors to relay messages. There are three critical issues for the sensors network, connectivity, power consuming and fault tolerence. If the network is not connected, it is splitted into several disjoint parts and each part can't communicate with each others. One method is to increase the signal power, but that increase power consuming. In this paper, the sensor network is modeled by the unit disc graph, the Bernoulli nodes, and the random point process on an unit area disc.

The unit disc graph is a widely used model. Each node has the same transmission range, a disc centered at that node. The Bernoulli node model is for the fault tolerance issue. Each node is either active or inactive with Bernoulli model. The probability that one node is active is p and the probability that one node is inactive is q ($p + q = 1$). There exists a direct link between two nodes if and only if both are active and their distance is less than the transmission radius. The random point process is used to model node distributing. $D = \{x \in \mathcal{R}^2 \mid |x| < \frac{1}{\sqrt{\pi}}\}$ is an unit area disc centering at the origin. There are n nodes distributed on D with independent and identical uniform distribution. Their positions are represented by $x_1, x_2, \cdots, x_n$, n random variables, and $x_i \neq x_j$ if $i \neq j$.

Connectivity, power consuming and fault tolerance are discussed together in this paper. If the transmission radius is a function of the number of Bernoulli nodes, what is the probability of the event that the random graph is connected? Here the number of nodes is denoted as n and the transmission range is denoted as r_n. If without causing confusion, we may only write r for r_n. We will show that if $\pi {r_n}^2 = \frac{\ln n + \xi}{pn}$, the induced random graph has good connectivity with the probability $e^{-(e^{-\xi})}$ for any real number ξ and $0 < p \leq 1$ as $n \to \infty$. $\Lambda(\xi) = e^{-(e^{-\xi})}$ is the Gumbel extreme-value function.

Relative problems are discussed in different models for different applications. The random graph problems [1] consider the properties of a graph with n nodes and k edges chosen randomly from the complete graph K_n. In the geometry random graph problems [2][3][4][5][6][7], the position is one of the key informations. The location of a nodes is a random variable and the existence of an edge between two nodes is dependent on the geometric information. Rectangle (cube) [2][3][4][5][6] and disc (ball) [4][7] are the two most popular bounded topologies on which the random variable are defined. If the random variable is defined on a bounded area, the probability distribution as points locating close to the boundary is different to the probability distribution as points locating in the interior area. This is boundary effects. There are several methods to handling boundary effects. Henze [2] given the asymptotic distribution of the maximal rth-nearest-neighbor on d-dimension cube and with point distributions. He avoided boundary effects by the delicate definition of rth-nearest-neighbor. Dette et al. [3][4] discussed the problem with uniform point distribution and extended the result to d-dimension cube and ball with directly handling boundary effects. They also shown that boundary effects depends on the space dimension and the topology. Penrose [5] applied toroidal metrics to avoid boundary effects for the longest edge of the minimal spanning tree problem. He [6] applied the relation between random point process and Poisson point process to solve the probability of k-connectivity. The result of continuum percolation [8][9][10][11] is applied to solve the connectivity problem [5][6][7].

Section II is about notations and basic ideas. The calculation of the asymptotic probability that the random graph is 1-degree is given in section III. Section IV give the relation between 1-degree property and connectivity. Section V is the conclusion and future works.

II. Preliminary and Notation

The problem is discussed on $\mathcal{R}^2$ space with L_2 metrics in this paper. $D_r(x) = \{y \mid |x - y| < r\}$ is the open disc with radius r and center x. $D = D_{\frac{1}{\sqrt{\pi}}}((0,0))$ is the unit area disc with center at the origin. $\{x_1, x_2, \cdots, x_n\} \subset D$ is an instance of n random points with identical, independent and uniform

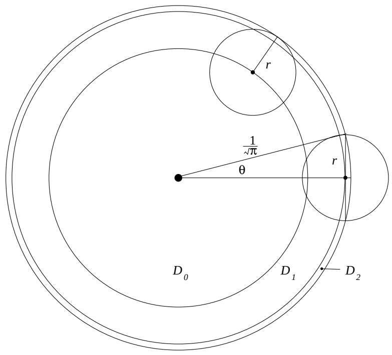

Fig. 1. Partitions of the unit area disc D

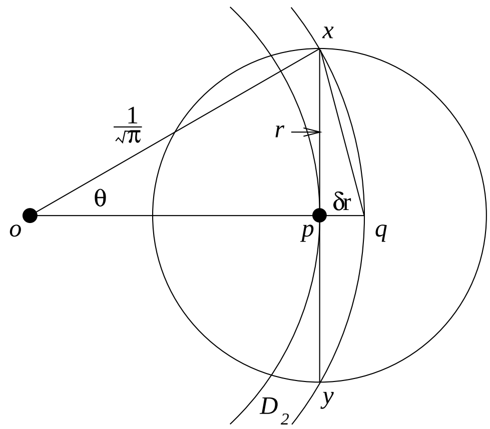

Fig. 2. How to calculate δr

distribution on D. Each one of them represents the location of a sensor with same transmission radius. The transmission radius, denoted by r_n, is a function of n. If without raising ambiguity, it may just be written as r. Each node is either active or inactive with probability. The probability that one node is active is p and the probability that one node is inactive is q ($p + q = 1$). There exists a direct link between two nodes if and only if both are active and their distance is less than r. $\mathcal{G}(n, r, p)$ denotes the graph induced by an instance of n random points process with transmission radius r and Bernoulli node probability p. The vertex set is composed of all the active nodes and the edge set is composed of all the directly links.

D is divided into 3 disjoint regions, D_0, D_1 and D_2, according to the transmission radius r. See Fig.1. There are 3 circles all with same center but with different radii $\frac{1}{\sqrt{\pi}} - r$, $\frac{1}{\sqrt{\pi}} - \delta r$ and $\frac{1}{\sqrt{\pi}}$. The largest one is the boundary of D. The smallest one is scratched by the centers of circles with radius r and tangent at the boundary of D. The medium one is scratched by the centers of circles with radius r and whose two intersection points with the boundary of D constituting its diameter. See Fig.2 for the calculation of δr. o is the center of D and p is the center of the disc with radius r. x, y are the two intersection point of these two discs. Since $\overline{ox} = \overline{oq}$ and $\overline{oq} \perp \overline{xy}$, we can get $\angle pxq = \frac{1}{2}\angle xoq = \frac{1}{2}\theta$. So $\delta r = r \tan\frac{\theta}{2}$. If $0 \leq \theta \leq \frac{\pi}{2}$, i.e. $r \leq \frac{1}{\sqrt{\pi}}$, $\delta r \leq r \sin\theta = \frac{r^2}{\sqrt{\pi}}$. D_0 is the disc bounded by the smallest circle. D_1 is the ring between the smallest circle and the medium circle. D_2 is the ring between the medium circle and the largest circles. $\partial D = D_1 \cup D_2$ is the area outside of the D_0. The areas of these regions usually is estimated by

$$
\begin{aligned}
|D_0| &= \pi(\frac{1}{\sqrt{\pi}} - r)^2 \leq 1 \\
|\partial D| &\leq 2\sqrt{\pi} r \\
|D_1| &\leq 2\sqrt{\pi}(r - \delta r) \leq 2\sqrt{\pi} r \\
|D_2| &\leq 2\sqrt{\pi}\delta r \leq 2r^2 \text{ (as } r \text{ is small enough)}
\end{aligned}
$$

The methodology for the extreme-value problem [12] used by Henze and Dette in [2][3][4] is adapted to find the asymptotic probability that $\mathcal{G}(n, r_n, p)$ is an 1-degree graph. Since $x_1, x_2, \cdots, x_n$ are random variables with iid, then for any $\{x_{s_1}, x_{s_2}, \cdots, x_{s_k}\} \subseteq \{1, 2, \cdots n\}$

$$
\begin{aligned}
& \Pr(x_1, \cdots, x_k \text{ are isolated points}) \\
= \; & \Pr(x_{s_1}, \cdots, x_{s_k} \text{ are isolated points})
\end{aligned}
$$

Using the inclusion-exclusion principle, we have

$$
\begin{aligned}
& \Pr(\mathcal{G}(n, r_n, p) \text{ is 1-degree}) \\
= \; & 1 - \Pr(\text{there exists at least one isolated point}) \\
= \; & 1 + \sum_{k=1}^{n} (-1)^k C_k^n \Pr(x_1, \cdots, x_k \text{ are isolated points}) \\
= \; & 1 + \sum_{k=1}^{n} \frac{(-1)^k}{k!} P_k^n \Pr(x_1, \cdots, x_k \text{ are isolated points})
\end{aligned}
$$

For any real number ξ, $0 < \xi < \infty$, there exists $r_n(\xi)$ such that for all integers $1 \leq k \leq n$

$$
\lim_{n\to\infty} n^k \Pr(x_1, \cdots, x_k \text{ are isolated points}) = e^{-k\xi}
$$

, then

$$
\lim_{n\to\infty} \Pr(\mathcal{G}(n, r_n(\xi), p) \text{ is 1-degree}) = e^{-(e^{-\xi})}
$$

The probability that $x_1, \cdots, x_k$ are isolated points would be an integral on $D^k = \{(x_1, \cdots, x_k) \mid x_1, \cdots, x_k \in D\}$, the k-fold Cartesian product of the unit area disk. Depending on the distance between $x_1, \cdots, x_k$, D^k will be splitted up. Give $x \in D^k$ and r_n, an equivalent relation is induced. x_i is equivalent to x_j, if and only if, either $i = j$ or there exists an integer sequence $(i_1, i_2, \cdots, i_m)$ such that $1 \leq i_1, \cdots, i_m \leq k$, $i_1 = i$, $i_m = j$, and $D_{r_n}(x_{i_s}) \cap D_{r_n}(x_{i_{s+1}}) \neq \emptyset$ for all $1 \leq s < m$. $l(x) = (l_1(x), l_2(x), \cdots, l_k(x))$ is a k-tuple. Here $l_i(x)$ is the number of equivalent classes with i elements. It is obvious that $\sum_{i=1}^{k} i l_i(x) = k, \forall x \in D^k$. Let $\mathcal{L}_k = \{(l_1, l_2, \cdots, l_k) \mid \sum_{i=1}^{k} i l_i = k\}$ and $D_{nk}(L \in \mathcal{L}_k) = \{x \in D^k \mid l(x) = L\}$. Then $D^k = \bigcup_{L \in \mathcal{L}_k} D_{nk}(L)$ and $D_{nk}(L_1) \cap D_{nk}(L_2) = \emptyset$ if $L_1 \neq L_2$. Let

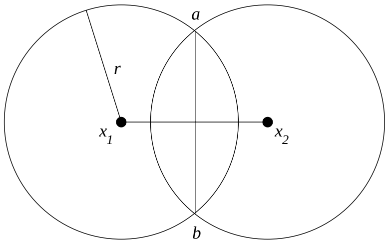

Fig. 3. Two intersected discs

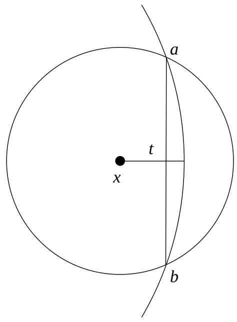

Fig. 4. Disc with center on D_1

L_0 to denote $(k, 0, \cdots, 0)$. Then $D_{nk}(L_0) = \{x \in D^k \mid |x_i - x_j| \geq 2r_n, \forall i \neq j\}$ is the space that $D_{r_n}(x_i) \cap D_{r_n}(x_j) = \emptyset$, $\forall i \neq j$.

For a given $L = (l_1, l_2, \cdots, l_k) \in \mathcal{L}_k$, $D_{nk}(L)$ can be further decomposed. We focus on those $x \in D_{nk}(L)$ that $\{x_1\}$, $\cdots$,$\{x_{l_1}\}$ form all l_1's 1-element equivalent classes, $\{x_{l_1+1}, x_{l_1+2}\}$, $\cdots$, $\{x_{l_1+2(l_2-1)+1}, x_{l_1+2(l_1-1)+2}\}$ form all l_2's 2-element equivalent classes, and so on. Let $\widetilde{D}_{nk}(L)$.denote the space formed by this kind of x. Suppose $f(x)$ is a function symmetric to all axes, then

$$\int_{x \in D_{nk}(L)} f(x)dx = \frac{k!}{\prod_{i=1}^{k}(i!)^{l_i} l_i!} \int_{x \in \widetilde{D}_{nk}(L)} f(x)dx$$

The calculation of probability is related to the measure of discs. $A_1(x \in D) = |D_{r_n}(x) \cap D|$ is the area of $D_{r_n}(x)$ within D, and $A_k(x \in D^k) = |\bigcup_{i=1}^{k} D_{r_n}(x_i) \cap D|$ is the area of $\bigcup_{i=1}^{k} D_{r_n}(x_i)$ within D. The following lemma is for estimation of those areas.

Lemma 1: If r is small enough, then

1) If $t = |x_2 - x_1| \leq 2r$, there exists a constant c such that $|D_r(x_1) \cup D_r(x_2)| \geq \pi r^2 + crt$.
2) If $x \in D_1$ and t ($t \in [\delta r, r]$) is the distance from x to the boundary of D, there exists a constant c such that $A_1(x) \geq \frac{1}{2}\pi r^2 + crt$.
3) If $x_1 \in D_0$, $x_2 \in D$ and $t = |x_2 - x_1| \leq 2r$, there exists a constant c such that $A_2((x_1, x_2)) \geq A_1(x_1) + crt$.
4) If $x_1 \in \partial D$, $|x_2| \leq |x_1|$ and $t = |x_2 - x_1| \leq 2r$, there exists a constant c such that $A_2((x_1, x_2)) \geq A_1(x_1) + crt$.

Proof: **(Part 1)** See Fig. 3. Consider $f_1(t) = |D_r(x_1) \backslash D_r(x_2)|$. Since $f_1^{'}(t) = |a - b| = 2\sqrt{r^2 - (\frac{t}{2})^2}$ is decreasing if $t \in [0, 2r]$, $f_1(t)$ is convex. And $f_1(0) = 0$ and $f_1(2r) = \pi r^2$, so let $c = \frac{\pi}{2}$. This is proved.

(Part 2) See Fig. 4. Consider $f_2(t) = A_1(x)$. $f_2^{'}(t) = |a - b|$ is decreasing if $t \in [\delta r, r]$, $f_2(t)$ is convex if $x \in D_1$. Let $c = 1$. Since

$$\begin{cases} f_2(\delta r) > \frac{1}{2}\pi r^2 + r\delta r \\ f_2(r) = \pi r^2 > \frac{1}{2}\pi r^2 + r^2 \end{cases}$$

So this is proved.

(Part 3) If $x_2 \in D_0$, it is reduced to **(Part 1)**. If $x_2 \in \partial D$, consider the worst case, i.e. x_1 is on the boundary of D_0 and x_2 is as close to the boundary of D as possible. Let $f_3(t = |x_2 - x_1|) = \min_{t=|x_2-x_1|} |(D_r(x_2) \backslash D_r(x_1)) \cap D|$. If $t \in [0, r]$, $f_3(t)$ increase and then decrease and $f_3(r) = \frac{\sqrt{3}}{2}r^2 - \frac{1}{6}\pi r^2 - c_1(r)$. If $t \in [r, 2r]$, $f_3(t)$ is a convex function since $f_3(t) = f_1(t) - \frac{1}{2}\pi r^2 - c_1(r)$. Here $\lim_{r \to 0} c_1(r) = 0$. Let $c = \min(\frac{\sqrt{3}}{2} - \frac{1}{6}\pi, \frac{1}{2}\pi) - \varepsilon$. So this is proved.

(Part 4) For a given x_1, the minimal value of $|(D_r(x_1) \cup D_r(x_2)) \cap D|$ happens as $|x_2| = |x_1|$. When r is small enough, the area of $(D_r(x_2) \backslash D_r(x_1)) \backslash D$ is just a little more than half of $D_r(x_2) \backslash D_r(x_1)$. Let $c = \frac{\pi}{4} - \varepsilon$. This is proved. ■

The following inequality [12][2] is useful to estimate the upper bound and lower bound.

$$e^{-mz} - (1 - z)^m(e^{-2mz^2} - 1) < (1 - z)^m \leq e^{-mz} \tag{1}$$

It works for all integer m and real number $0 < z < \frac{1}{2}$.

In a n-point random point process, we use the following notations for convenience. N_i is the number of x_i's neighbors. $X_k = \{x_1, \cdots, x_k$ don't have active neighbors$\}$. $Y_k = \{x_1, \cdots, x_k$ from an equivalent class$\}$. $Z_{ij} = \{x_i$ has j neighbors and all are inactive$\}$. $\overline{Y}_k = \{x_1, \cdots, x_k$ form a k-element connected component$\}$. $E_k = \{x \in D^k \mid x_1, \cdots, x_k$ form an equivalent class$\}$. $\overline{E}_k = \{x \in D^k \mid x_1, \cdots, x_k$ form a k-element connected component$\}$. In those notations, n is omitted.

III. Asymptotic Probability of 1-Degree

In this section, the probability that the random graph is 1-degree is given. With the Bernoulli node model, each node is active or inactive. The active nodes should dense enough such that $\mathcal{G}(n, r_n, p)$ is 1-degree and each inactive node also has at least one active neighbor. So $\mathcal{G}(n, r_n, p)$ can always keep 1-degree even when an inactive node turns to active. The major goal here is to figure out the probability

$$\Pr(\text{each node has at least one active neighbor})$$

The detail calculation, following the outline in Section II, is given. The proof is divided into three steps, $n \Pr(x_1$ doesn't have active neighbors), $n^2 \Pr(x_1, x_2$ don't have active neighbors), and $n^k \Pr(x_1, \cdots, x_k$ don't have active neighbors).

Lemma 2: $x_1, x_2, \cdots, x_n$ are n random points with uniform distribution on the unit area disc D. r_n is the transmission radius. Each nodes independently associates with success probability p. Here p, ξ are fixed real numbers and $0 < p \leq 1$. Let $\pi {r_n}^2 = \frac{\ln n + \xi}{pn}$, then

$$\lim_{n\to\infty} n \Pr(x_1 \text{ doesn't have active neighbors}) = e^{-\xi}$$

Proof: x_1 might have neighbors but all of them must be inactive. The probability can be calculated by the number of x_1's neighbors.

$$n \sum_{i=0}^{n-1} \Pr(\text{all neighbors are inactive} \mid N_1 = i) \Pr(N_1 = i)$$

$\Pr(N_1 = i)$ is equal to $\int_D C_i^{n-1}(1 - A_1(x))^{n-1-i} A_1(x)^i dx$ and $\Pr(\text{all neighbors are inactive} \mid N_1 = i)$ is equal to q^i. So

$$\begin{aligned}
& n \Pr(x_1 \text{ doesn't have active neighbors}) \\
= \;& n \sum_{i=0}^{n-1} q^i \int_D C_i^{n-1}(1 - A_1(x))^{n-1-i} A_1(x)^i dx \\
= \;& n \int_D \sum_{i=0}^{n-1} C_i^{n-1}(1 - A_1(x))^{n-1-i} (A_1(x) q)^i dx \\
= \;& n \int_D (1 - A_1(x) + qA_1(x))^{n-1} dx \\
= \;& n \int_D (1 - pA_1(x))^{n-1} dx
\end{aligned}$$

Depending on the location of x_1, the probability is calculated on D_0, D_1 and D_2. On D_0, using inequality 1 and

$$\begin{aligned}
n \int_{x \in D_0} e^{-np\pi {r_n}^2} dx &= ne^{-(\ln n + \xi)} (\pi (\frac{1}{\sqrt{\pi}} - r_n)^2) \\
&= e^{-\xi} (\pi (\frac{1}{\sqrt{\pi}} - r_n)^2)
\end{aligned}$$

Then we have

$$\begin{aligned}
& \lim_{n\to\infty} n \int_{x \in D_0} (1 - pA_1(x))^{n-1} dx \\
= \;& \lim_{n\to\infty} n \int_{x \in D_0} e^{-np\pi {r_n}^2} dx \\
= \;& e^{-\xi}
\end{aligned}$$

On D_1, apply Lemma 1 and Eq. 1

$$\begin{aligned}
& n \int_{x \in D_1} (1 - pA_1(x))^{n-1} dx \\
\leq \;& \frac{1}{1 - p\pi r_n^2} n \int_{t=\delta r_n}^{r_n} e^{-np(\frac{1}{2}\pi {r_n}^2 + cr_n t)} 2\sqrt{\pi} dt \\
\leq \;& \frac{2\sqrt{\pi}}{1 - p\pi r_n^2} ne^{-\frac{1}{2}pn\pi {r_n}^2} (\frac{1}{cpnr_n} (1 - e^{-cpnr_n{}^2})) \\
= \;& \frac{2\pi(1 - e^{-cpnr_n{}^2})}{(1 - p\pi {r_n}^2) c \sqrt{pe^{\xi}}} \frac{1}{\sqrt{(\ln n + \xi)}} \\
= \;& O(\frac{1}{\sqrt{\ln n}}) \\
\to \;& 0 \text{ as } n \to \infty
\end{aligned}$$

On D_2, $A_1(x) \geq (\frac{1}{2} - c_1(r_n)) \pi {r_n}^2$ and $\lim_{r_n \to 0} c_1(r_n) = 0$.

$$\begin{aligned}
& n \int_{x \in D_2} (1 - pA_1(x))^{n-1} dx \\
\leq \;& \frac{1}{1 - p\pi {r_n}^2} ne^{-(\frac{1}{2} - c_1(r_n)) np\pi {r_n}^2} (2\sqrt{\pi} \delta r_n) \\
= \;& \frac{2}{1 - p\pi {r_n}^2} ne^{-(\frac{1}{2} - c_1(r_n)) pn\pi {r_n}^2} {r_n}^2 \\
= \;& \frac{2}{(1 - p\pi {r_n}^2) e^{(\frac{1}{2} - c_1(r_n))\xi} \pi p} \frac{\ln n + \xi}{n^{(\frac{1}{2} - c(r_n))}} \\
\to \;& 0 \text{ as } n \to \infty
\end{aligned}$$

Combining these results, Lemma is proved. ■

The next step is to consider the case in which both x_1, x_2 don't have active neighbors..

Lemma 3: $x_1, x_2, \cdots, x_n$ are n random points with uniform distribution on the unit area disc D. r_n is the transmission radius. Each nodes independently associates with success probability p. Here p, ξ are fixed real numbers and $0 < p \leq 1$. Let $\pi {r_n}^2 = \frac{\ln n + \xi}{pn}$, then

$$\lim_{n\to\infty} n^2 \Pr(x_1, x_2 \text{ don't have active neighbors}) = e^{-2\xi}$$

Proof: The probability is calculated in two cases depending on the distance between x_1 and x_2.

$$\begin{aligned}
& n^2 \Pr(X_2) \\
= \;& n^2 \Pr(X_2 \cap \{\overline{x_1 x_2} \geq 2r_n\}) + n^2 \Pr(X_2 \cap \{\overline{x_1 x_2} < 2r_n\})
\end{aligned}$$

First, we show the second term tends to 0 as $n \to \infty$. Let $A_{1\setminus 2}(x) = |(D_{r_n}(x_1) \setminus D_{r_n}(x_2)) \cap D|$, $A_{2\setminus 1}(x) = |(D_{r_n}(x_2) \setminus D_{r_n}(x_1)) \cap D|$ and $A_{1\cap 2}(x) = |(D_{r_n}(x_1) \cap D_{r_n}(x_2)) \cap D|$. It is obvious that $A_2(x) = A_{1\setminus 2}(x) + A_{2\setminus 1}(x) + A_{1\cap 2}(x)$. Let i ,j and k denote the number of nodes excepting x_1, x_2 locating at $A_{1\setminus 2}(x)$, $A_{2\setminus 1}(x)$ and $A_{1\cap 2}(x)$. If $r_n \leq \overline{x_1 x_2} < 2r_n$, the probability $\Pr(X \cap \{r_n \leq \overline{x_1 x_2} < 2r_n\})$ is equal to

$$\begin{aligned}
& \int_{\substack{x \in D^2 \\ r_n \leq |x_2 - x_1| < 2r_n}} dx \sum_{i+j+k=0}^{n-2} C_i^{n-2} C_j^{n-2-i} C_k^{n-2-i-j} \\
& (1 - A_2(x))^{n-2-i-j} A_{1\setminus 2}(x)^i A_{2\setminus 1}(x)^j A_{1\cap 2}(x)^k q^{i+j+k}
\end{aligned}$$

If $\overline{x_1x_2} < r_n$, both x_1, x_2 must be inactive and the probability $\Pr(X \cap \{\overline{x_1x_2} < r_n\})$ is equal to

$$\int_{\substack{x\in D^2\\|x_2-x_1|<r_n}} dx \sum_{i+j+k=0}^{n-2} C_i^{n-2} C_j^{n-2-i} C_k^{n-2-i-j} q^2$$
$$(1-A_2(x))^{n-2-i-j} A_{1\backslash 2}(x)^i A_{2\backslash 1}(x)^j A_{1\cap 2}(x)^k q^{i+j+k}$$

Combine these two result and after straightforward calculation, we have

$$\Pr(X_2 \cap \{\overline{x_1x_2} < 2r_n\}) \le \int_{\substack{x\in D^2\\|x_2-x_1|<2r_n}} (1-pA_2(x))^{n-2} dx$$

Consider the boundary effect, this integral can be evaluated in two cases depending on $x_1 \in D_0$ or $x_1 \in \partial D$

$$\begin{aligned} & n^2 \Pr(X_2 \cap \{\overline{x_1x_2} < 2r_n\}) \\ \le\ & n^2 \int_{\substack{x\in D^2\\|x_2-x_1|<2r_n}} (1-pA_2(x))^{n-2} dx \\ \le\ & C(F_1 + 2F_2) \end{aligned}$$

Here $C = \frac{1}{(1-2p\pi r_n{}^2)^2}$ and

$$\begin{aligned} F_1 &= n^2 \int_{\substack{|x_2-x_1|<2r_n\\x_1\in D_0, x_2\in D}} e^{-npA_2(x)} dx \\ F_2 &= n^2 \int_{\substack{|x_2-x_1|<2r_n\\x_1\in\partial D, |x_2|\le|x_1|}} e^{-npA_2(x)} dx \end{aligned}$$

For F_1, apply $A_2(x) \ge \pi r_n{}^2 + cr_n|x_2-x_1|$ (Lemma 1)

$$\begin{aligned} & n^2 \int_{\substack{|x_2-x_1|<2r_n\\x_1\in D_0, x_2\in D}} e^{-npA_2(x)} dx \\ \le\ & n^2(\pi(\frac{1}{\sqrt{\pi}} - r_n)^2)(e^{-p_1 n r_n{}^2} \int_{t=0}^{2r_n} e^{-cp_1 n r_n t} 2\pi t dt) \\ =\ & O(\ln^{-1} n) \end{aligned}$$

For F_2, apply $A_2(x) \ge A_1(x_1) + cr_n|x_2-x_1|$ (Lemma 1)

$$\begin{aligned} & n^2 \int_{\substack{|x_2-x_1|<2r_n\\x_1\in\partial D, |x_2|\le|x_1|}} e^{-npA_2(x)} dx \\ \le\ & n^2 \int_{\substack{|x_2-x_1|<2r_n\\x_1\in\partial D, |x_2|\le|x_1|}} e^{-np(A_1(x_1)+cr_n|x_2-x_1|)} dx_1 dx_2 \\ \le\ & (n\int_{x_1\in\partial D} e^{-npA_1(x_1)} dx_1)(n\int_{t=0}^{2r_n} e^{-npcr_n t} \pi t dt) \\ =\ & O(\ln^{-\frac{3}{2}} n) \end{aligned}$$

We still need to show that $n^2 \Pr(X \cap \{\overline{x_1x_2} \ge 2r_n\})$ tends to $e^{-2\xi}$ as $n \to \infty$. Let i ,j denote the number of nodes located in $A_1(x_1)$ and $A_1(x_2)$.

$$\begin{aligned} & \Pr(X_2 \cap \{\overline{x_1x_2} \ge 2r_n\}) \\ =\ & \int_{x\in D_{n2}(L_0)} dx \sum_{i+j=0}^{n-2} C_i^{n-2} C_j^{n-2-j} \\ & (1-A_2(x))^{n-2-i-j} (qA_1(x_1))^i (q_1 A_1(x_2))^j \\ =\ & \int_{x\in D_{n2}(L_0)} (1-p(A_1(x_1)+A_1(x_2)))^{n-2} dx \end{aligned}$$

Apply Eq. 1 and $A_1(x_1) + A_1(x_2) \ge A_2(x)$, then

$$\begin{aligned} & n^2 \Pr(X \cap \{\overline{x_1x_2} \ge 2r_n\}) \\ \sim\ & n^2 \int_{x\in D^2} e^{-pn(A_1(x_1)+A_1(x_2))} dx \\ =\ & (n\int_{x\in D} e^{-pnA_1(x)} dx)^2 \\ \sim\ & (n \Pr(x_1 \text{ doesn't have active neighbors}))^2 \\ =\ & e^{-2\xi} \text{ as } n \to \infty \end{aligned}$$

Lemma is proved. ■

The last part is to prove the general case $k \ge 3$. The argument here also works for the case $k = 2$.

Lemma 4: $x_1, x_2, \cdots, x_n$ are n random points with uniform distribution on the unit area disc D. r_n is the transmission radius. Each nodes independently associates with success probability p. Here p, ξ are fixed real numbers and $0 < p \le 1$. Let $\pi r_n{}^2 = \frac{\ln n + \xi}{pn}$, then $\forall k \ge 2$

$$\lim_{n\to\infty} n^k \Pr(x_1, \cdots, x_k \text{ don't have active neighbors}) = e^{-k\xi}$$

Proof: The probability is discussed in two terms. If n goes to infinite, the first term tends to $e^{-k\xi}$ and the second term tends to 0.

$$\begin{aligned} & n^k \Pr(X_k) \\ =\ & n^k \Pr(X_k \cap \{x \in D_{nk}(L_0)\}) \\ & + n^k \Pr(X_k \cap \{x \in D^k \backslash D_{nk}(L_0)\}) \end{aligned}$$

In Lemma 5, we will show that for any $k \ge 2$

$$\lim_{n\to\infty} n^k \Pr(X_k \cap Y_k) = 0$$

For any fix $k \ge 2$ and $L \in \mathcal{L}_k \backslash \{L_0\}$, apply this result, then

$$\begin{aligned} & n^k \Pr(X_k \cap \{x \in D_{nk}(L)\}) \\ =\ & \frac{k!}{\prod_{i=1}^k (i!)^{l_i} l_i!} n^k \Pr(X_k \cap \{x \in \widetilde{D}_{nk}(L)\}) \\ \le\ & \frac{k!}{\prod_{i=1}^k (i!)^{l_i} l_i!} \prod_{i=1}^k (n^i \Pr(X_i \cap Y_i))^{l_i} \\ =\ & 0 \text{ as } n \to \infty \text{ and } L \ne L_0 \end{aligned}$$

The next is to find out $\lim_{n\to\infty} n^k \Pr(X_k \cap \{x \in D_{nk}(L_0)\})$.

$$
\begin{aligned}
& n^k \Pr(X_k \cap \{x \in D_{nk}(L_0)\}) \\
= \;& n^k \sum_{N_1+N_2+\cdots N_k=0}^{n-k} \Pr(\bigcap_{i=1}^{k} Z_{iN_i} \cap \{x \in D_{nk}(L_0)\}) \\
= \;& n^k \int_{x\in D_{nk}(L_0)} \sum_{N_1+N_2+\cdots N_k=0}^{n-k} (1-A_k(x))^{n-k-\sum_{i=1}^{k} N_i} \\
& \prod_{i=1}^{k} (C_{N_i}^{\,n-k-\sum_{j=1}^{i-1} N_j} (A_1(x_i)q)^{N_i}) dx \\
= \;& n^k \int_{x\in D_{nk}(L_0)} (1-p\sum_{i=1}^{k} A_1(x_i))^{n-k} dx
\end{aligned}
$$

Since $\sum_{i=1}^{k} A_1(x_i) \geq A_k(x)$ and using previous result, we can get

$$
\lim_{n\to\infty} n^k \int_{x\in D^k \backslash D_{nk}(L_0)} (1-p_1\sum_{i=1}^{k} A_1(x_i))^{n-k} dx = 0
$$

And using Eq. 1, then

$$
\begin{aligned}
& n^k \Pr(X_k \cap \{x \in D_{nk}(L_0)\}) \\
\sim \;& n^k \int_{x\in D_{nk}(L_0)} e^{-p_1 n \sum_{i=1}^{k} A_i(x)} dx \\
\sim \;& (n \int_{x\in D} e^{-p_1 n A_1(x)} dx)^k \\
\sim \;& (n \Pr(x_1 \text{ doesn't have active neighbors}))^k \\
= \;& e^{-k\xi} \text{ as } n \to \infty
\end{aligned}
$$

Lemma is proved. ■

The last lemma is to give the probability under the condition that $x_1, x_2, \cdots, x_k$ form an equivalent class.

Lemma 5: $x_1, x_2, \cdots, x_n$ are n random points with uniform distribution on the unit area disc D. r_n is the transmission radius. Each nodes independently associates with success probability p. Here p, ξ are fixed real numbers and $0 < p \leq 1$. Let $\pi {r_n}^2 = \frac{\ln n+\xi}{pn}$. Then for any fixed integer $k \geq 2$

$$
\lim_{n\to\infty} n^k \Pr(X_k \cap Y_k) = 0
$$

Proof: Using similar argument in Lemma 3, we can get

$$
n^k \Pr(X_k \cap Y_k) \leq n^k \int_{x\in E_k} (1-pA_k(x))^{n-k} dx
$$

E_k is going to be splitted up. Let $M = \{0,1,2\}, m \in M^k$, and $E_k(m) = \{x \in E_k \mid x_i \in D_{m_i}\}$, then $E_k = \bigcup_{m\in M^k} E_k(m)$. For a given m, let $m_{\max}$ and $m_{\min}$ denote $\max_{1\leq i\leq k} m_i$ and $\min_{1\leq i\leq k} m_i$. We can get $|E_k(m)| \leq O({r_n}^{m_{\max}} (\pi {r_n}^2)^{k-1})$ and $A_k(x) \geq 2^{-m_{\min}} \pi {r_n}^2$. After straight forward calculation,

$$
\begin{aligned}
& n^k \int_{x\in E_k(m)} (1-pA_k(x))^{n-k} dx \\
\leq \;& O(n^{1-(2^{-m_{\min}}+\frac{m_{\max}}{2})} \ln^{k-1+\frac{m_{\max}}{2}} n)
\end{aligned}
$$

If $m_{\max} > m_{\min}$ or $(m_{\max}, m_{\min}) = (2,2)$, the probability tends to 0 as $n \to \infty$. For the other two conditions, i.e. $(m_{\max}, m_{\min})$ is equal to $(0,0)$ or $(1,1)$, the probability can be estimated more tightly. $(m_{\max}, m_{\min}) = (0,0)$ means $m = (0,\cdots,0)$. Let $\eta_{nk} = (\frac{k \ln\ln n}{\ln n})^{\frac{1}{2}}$, $S_1 = \{x \in E_k((0,\cdots,0)) \mid \exists i \neq j \text{ s.t. } |x_i - x_j| \geq \eta_{nk} r_n\}$ and $S_2 = D \backslash S_1$. If $x \in S_1$, we have $A_k(x) \geq (1+\frac{k \ln\ln n}{\ln n}) \pi r_n^2$, $e^{-npA_k(x)} \leq O(\frac{1}{n \ln^k n})$ and $|S_1| = O((\frac{\ln n}{n})^{k-1})$. Then

$$
\begin{aligned}
& n^k \int_{x\in S_1} (1-pA_k(x))^{n-k} dx \\
\leq \;& O(\ln^{-1} n) \\
= \;& 0 \text{ as } n \to \infty
\end{aligned}
$$

If $x \in S_2$, it means $|x_i - x_j| < \eta_{nk} r_n$ for $\forall i \neq j$.

$$
\begin{aligned}
& n^k \int_{x\in S_2} (1-pA_k(x))^{n-k} dx \\
\leq \;& O((\ln\ln n)^{k-2}) n^2 \int_{\substack{x\in D_0^2 \\ |x_2-x_1|<\eta_{nk} r_n}} (1-pA_2(x))^{n-k} dx \\
= \;& O((\ln\ln n)^{k-2} \ln^{-1} n) \\
= \;& 0 \text{ as } n \to \infty
\end{aligned}
$$

Similar argument can apply to the case $(m_{\max}, m_{\min}) = (1,1)$. So this is proved. ■

Theorem 6: $x_1, x_2, \cdots, x_n$ are n random points with uniform distribution on the unit area disc D. r_n is the transmission radius. Each nodes independently associates with success probability p. Here p, ξ are fixed real numbers and $0 < p \leq 1$. Let $\pi {r_n}^2 = \frac{\ln n+\xi}{pn}$. Then

$$
\lim_{n\to\infty} \Pr(\text{each node has at least one active neighbor}) = e^{-(e^{-\xi})}
$$

Proof: Theorem follows Lemma 2, 3, 4. ■

IV. Connectivity in $G(n, r_n, p)$

In this section, the connectivity of $G(n, r_n, p)$ is the major concern. We show that the probability that there exists k-element connected component ($k \geq 2$) in $G(n, r_n, p)$ tends to 0 as $n \to \infty$. So one node is either isolated or belongs to a connected component with cardinality tends to infinite as $n \to \infty$.

Theorem 7: If U is connected component in $G(n, r_n, p)$ with $\pi {r_n}^2 = \frac{\ln n + \xi}{pn}$, $Card(U)$ is either 1 or ∞ as $n \to \infty$.

Proof: Suppose $Card(U) \neq 1$. If $k \geq 2$ is a fixed integer,

$$\begin{aligned}
& \Pr(\{\exists U \text{ s.t. } Card(U) = k\}) \\
\leq \quad & C_k^n \Pr(W_k) \\
\leq \quad & n^k \int_{x \in \overline{E}_k} \sum_{i=0}^{n-k} C_i^{n-k} (A_k(x)q)^i (1 - A_k(x))^{n-k-i} dx \\
= \quad & n^k \int_{x \in \overline{E}_k} (1 - pA_k(x))^{n-k} dx
\end{aligned}$$

Use the result in Lemma 5 and $\overline{E}_k \subset E_k$, then $\forall k \geq 2$

$$\lim_{n \to \infty} \Pr(\{\exists U \text{ s.t. } Card(U) = k\}) = 0$$

Lemma is proved. ■

As n tends to ∞ and $G(n, r_n, p)$ is 1-degree, then there are no isolated points in $G(n, r_n, p)$ and the cardinality of connected components tends to infinite.

V. CONCLUSION

The fault tolerance of sensor networks is investigate by the probability of node failure. Let $\pi {r_n}^2 = \frac{\ln n + \xi}{pn}$. Then as $n \to \infty$, the probability of each nodes has at least one active node is $e^{-(e^{-\xi})}$ and the cardinality of a connected component is either 1 or tending to ∞. We believe that $G(n, r_n, p)$ is almost sure connected if $G(n, r_n, p)$ is 1-degree. But this still needs to be proved. Beside the node failure model, other failure models are also interesting. In the link failure model, we also have similar result.

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