Point placement for meshless methods using Sphere packing and Advancing Front methods

Xiang-Yang Li, Shang-Hua Teng, Alper Üngör
Department of Computer Science,
University of Illinois at Urbana-Champaign,
{xli2,steng,ungor}@cs.uiuc.edu

Summary

For simulation of numerical problems, meshless methods have emerged as an alternative to mesh based methods and become popular due to several reasons. Primarily, mesh generation is a difficult problem, Also these methods have generated promising results in the accuracy of the simulations. Similar to mesh generation problems, meshless methods also induce challenging geometric problems. Partition of Unity Method (PUM) is one of the meshless methods that is based on the definition of overlapping patches covering the domain. Some overlap among the patches is required. These patches should overlap to cover the domain, however, it should not be to the extend that deteriorates the accuracy. Also any single point in the domain should not be covered by more than a certain number of patches. We formalize this criteria to define a good point set for meshless methods in the second section the paper.

Earlier implementations employed arbitrary point set generation which could not support required structure of PUM. The alternative ”mesh free” approach uses a mesh to obtain the point set.

Biting Method is an advancing front technique that generates a good sphere packing [6]. The centers of the spheres, a well-spaced point set, can in turn be used to generate a provably good simplicial mesh. In this study, we use this approach to generate a good point set for meshless methods. However, instead of relating the radii of the spheres with the mesh spacing function, size of the spheres (patches) is defined by the support function and the overlap criteria described above. We prove that biting method can be used to generate a good point set for meshless methods.

Introduction

Meshless Methods Due to the diffculty of the mesh generation prolbem, alternative approaches for simulation of the numerical problems are sought. Meshless methods eliminates the need for a mesh structure, attempts to solve the numerical problems by constructing the approximation on the node based patches covering the domain. During the last decade, several of meshless methods are proposed including Partition of Unity Method [1] and element-free Galerkin method [2]. See Belytschko *et al.* for an excellent review of the meshless methods.

Previous Approaches Despite the variety of the meshless methods, point set generation problem did not take enough consideration. Indeed many of the implementations employ mesh generation packages, and use the point set of a mesh. Recent studies to generate point set for the meshless methods include octree based approach by Klaas and Shephard [5]. They use a level one adjusted octree, i.e. the level difference between terminal octants and their neighbors is no more than one, partition of the domain. Corner nodes of the octants defines the point set together with the patches defined by the sum of all octants sharing a particular corner node. Another approach by Choi and Kim [4] uses Voronoi diagram and weighted buble packing. Our approach is similar to the second one in using spherical elements to define the patches.

Biting Method Introduced by Li, Teng, and Üngör a provably good simplicial mesh generation algorithm, [6, 7, 8]. It uses an advancing front strategy to generate a good sphere packing. The centers of the spheres, a well-spaced point set, can in turn be used to generate a provably good tetrahedral mesh. In this study, we use this approach to generate a good point set for meshless methods. However, instead of relating the radii of the spheres with the mesh spacing function, size of the spheres (patches) is defined by the support function and the overlap criteria described above.

Outline The following section defines the problem formally. Next an algorithm is given to generate a good points set for the meshless methods. The quality of the point set is proved in the fifth section.

Problem Definition

Quality of the elements is very critical for numerical methods as it has huge impact on the accuracy of the simulation. The shape and size criteria are well studied for mesh based methods. However, there is no widely agreed criteria on the quality of a point set for the meshless methods. Various geometric shapes (including spheres, rectangles, simple polygons) are used as the patches attached to each point describing the domain of influence for that point. The size and shape of the patches together with the amount of overlap between the patches effects the accuracy of the numerical simulation. Following definition formalizes the criteria for a good point set. For proximity reasons, the domain of influence shape (patch shape) is chosen to be sphere. For flow problems, where anisotropy in the domain is important, this definition can be modified to use ellipse shaped patches.

Definition [Good Point Set for Meshless Methods] Given a domain Ω, we call a set of points P, *Good Point Set for Meshless Methods* if it satisfies the following criteria:

1. Each point $x \in P$ is associated with a sphere centered on $\mathbf{x}$ and have a radius related to the size of the support function at $\mathbf{x}$. Let S be the set of spheres corresponding to the points in P.
2. The spheres in S cover the whole domain Ω, i.e. any point x in the domain D is contained in at least one sphere in S.
3. The size of the intersection between two overlapping spheres in S is bounded from above by a constant factor of the size of the smaller sphere.
4. Any single point in the domain Ω should not be covered by more than a certain number of spheres in S.

The size of the domain of influence differs through out the domain due to numerical accuracy or domain geometry. We use a spacing function to denote how big a patch should be at a particular point in the domain. This is analagous to the global spacing function used in the meshing version of the biting algorithm [6].

Definition [Spacing Function] Let $\mathbf{x}$ be a point in the domain Ω, spacing function, $f(\mathbf{x})$, denotes the size of the domain of influence at $\mathbf{x}$.

The following assumption about how fast this function changes from point to point in the domain provides the theoretical base for our claims.

Definition [Lipschitz Property] A function $f()$ is *Lipschitz with a coefficiency* α if for any two points $\mathbf{x}$, $\boldsymbol{y}$ in the domain, $|f(\mathbf{x}) - f(\boldsymbol{y})| \leq \alpha||\mathbf{x} - \boldsymbol{y}||$.

We assume that given spacing function is α-Lipschitz for some constant α. Now the question is how to generate such a theoretically good point set.

Algorithm

This section adapts the *Biting Method* to generate a good point set for the meshless methods. Originally biting method is designed to generate a sphere packing, which in turn is used to generate a well-shaped mesh, i.e. the circumradius-to-shortest-edge ratio of every element in the mesh is bounded from above by a constant. It uses two types of spheres, biting spheres and packing spheres. They are centered at the same points $\mathbf{x}$, and have different radii: biting sphere has radius $C_b f(\mathbf{x})$, where C_b is the biting constant specified by the user of the algorithm. Packing sphere at $\mathbf{x}$ has radius $C_p f(\mathbf{x})$, where $C_p = \frac{C_b}{2+\alpha C_b}$. The reader is referred to [6, 7, 8] for detailed discussion of these two types of spheres. For mesh generation purposes, biting spheres are used to cover the domain thus for generating good sphere packing, packing spheres are used to provide the theoretical quality guarantee. In the actual biting method implementation, packing spheres are never created. For Meshless methods, the biting spheres centered on the generated point set are used to define the patches.

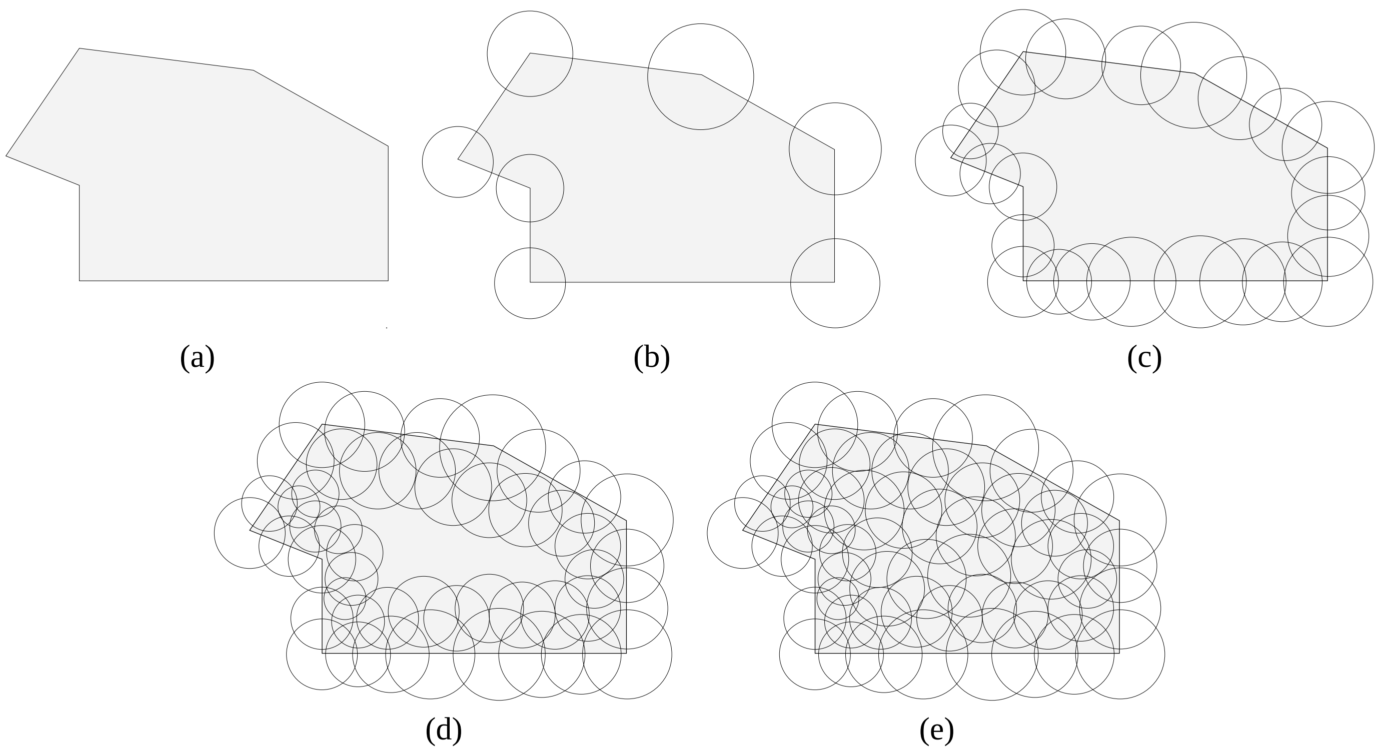

Figure 1: Illustration of the *biting* scheme: (a) initial PSLG domain; (b) bites only on the vertices of the polygon; (c) bites along the boundary; (d) biting of another layer towards the interior; (e) whole domain is covered by the biting spheres.

Algorithm `Biting`

1. Let the boundary of the domain be the initial front, see Figure 1 (a);
2. **[Points on the original Vertices]**: Bite all the input vertices by removing their biting spheres from the interior of the domain, see Figure 1 (b);

   Modify the front which becomes a set of segments and arcs. Segments are represented by the endpoints and arcs are represented by the center of the biting sphere.

3. **[Points on the Edges]**: Bite spheres centered on the input boundary: choose a vertex $\mathbf{x}$ on the front and remove its biting sphere. Whenever possible, we choose $\mathbf{x}$ on the intersection of some bitten spheres with the initial boundary, see Figure 1 (c).

   Modify the front by introducing the arc of the new biting spheres and removing the intersection of it with the front.

   Repeat until all initial input boundaries are bitten;

4. **[Points in the Interior of the Domain]**: Choose a vertex $\mathbf{x}$ on the front and remove its biting sphere, see Figure 1 (d) and (e);

   Modify the front by introducing the arc of the new biting sphere and removing the intersection of it with the front.

   Repeat until the advancing front is empty.

If the input is 3 dimensional domain, we need add one step after the 3^{rd} step: keep biting all biting speres centered on the initial input boundary faces, and modify the advancing front.

Guarantee for a Good Point Set

To provide the quality guarantee for the point set, we will discuss the achievement of the algorithm in terms of the quality criteria given in the second section. Due to the choice of the spacing function, the first criterion about the relation of the patches (spheres) with the point set is satisfied. Also it is very clear from the flow of the algorithm that there is no point in the domain that is not covered by at least one patch. For simplicity, let $B(\mathbf{x}, r)$ be the sphere centered at $\mathbf{x}$ with radius r. For the rest of the criteria we prove the following lemma.

Lemma 0.1 (Size of the Overlap) *The size of the overlapping region between any two spheres in S is bounded from above by a constant factor of the size of the smaller sphere.*

Proof: We prove the lemma by proving that for any two overlapped biting spheres centered at points $\mathbf{x}$ and $\boldsymbol{y}$, there is a constant factor of one sphere which is not covered by the other. For any two points $\mathbf{x}$ and $\boldsymbol{y}$ generated by the biting method, with no loss of generality suppose $\mathbf{x}$ is bitten before $\boldsymbol{y}$. If $C_b f(\mathbf{x}) \geq C_b f(\boldsymbol{y})$, then the proof is immediate since the intersection area would be less than half the size of either sphere. In the other case, we need to prove that the size of $B(\mathbf{x}, C_b f(\mathbf{x}))$ not covered by $B(\boldsymbol{y}, C_b f(\boldsymbol{y}))$ has a lower bound. Since $\mathbf{x}$ is bitten earlier than $\boldsymbol{y}$, $||\mathbf{x} - \boldsymbol{y}|| \geq C_b f(\mathbf{x})$.

Let $\boldsymbol{uv}$ be the diameter of $B(\mathbf{x}, C_b f(\mathbf{x}))$ on line $\mathbf{x}\boldsymbol{y}$, $\boldsymbol{u}$ being on the opposite side of $\boldsymbol{y}$. Multiplying the Lipschitz inequality by the constant C_b, we get $C_b f(\boldsymbol{y}) \leq C_b f(\mathbf{x}) + \alpha C_b ||\mathbf{x} - \boldsymbol{y}||$. The length of $\boldsymbol{uv}$ not covered by sphere $B(\boldsymbol{y}, c_b f(\boldsymbol{y}))$ is $||\mathbf{x} - \boldsymbol{y}|| + C_b f(\mathbf{x}) - C_b f(\boldsymbol{y})$, which is at least $(1 - \alpha C_b)||\mathbf{x} - \boldsymbol{y}|| \geq C_b(1 - \alpha C_b) f(\mathbf{x})$. Hence, at least $(\frac{1-\alpha C_b}{2})^2$ factor of the area of $B(\mathbf{x}, C_b f(\mathbf{x}))$ is not covered by $B(\boldsymbol{y}, C_b f(\boldsymbol{y}))$. □

Lemma 0.2 (Constant Ply) *Algorithm* `Biting` *generates a point set S, that has the constant ply property. In other words, there exists a constant k such that for every point $\boldsymbol{y} \in \Omega$, there are at most k biting balls that covers $\mathbf{y}$.*

Proof: We use the notion of packing spheres in order to prove this lemma. Packing spheres do not overlap and hence, there are only a constant number of packing spheres in a neihgborhood of a point in the domain. We consider any $\boldsymbol{y} \in \Omega$, and let $B(\mathbf{x}_1, C_b f(\mathbf{x}_1)), B(\mathbf{x}_2, C_b f(\mathbf{x}_2)), \cdots, B(\mathbf{x}_t, C_b f(\mathbf{x}_t))$ be the biting balls that cover $\boldsymbol{y}$. We assume that the ball $B(\mathbf{x}_i, C_b f(\mathbf{x}_i))$ is bitten before the ball $B(\mathbf{x}_{i+1}, C_b f(\mathbf{x}_{i+1}))$, for all $1 \leq i < t$.

As $\boldsymbol{y} \in B(\mathbf{x}_i, C_b f(\mathbf{x}_i))$, we have $||\mathbf{x}_i - \boldsymbol{y}|| \leq C_b f(\mathbf{x}_i)$. And $f()$ is α-Lipschitz implies that

$$-\alpha C_b f(\mathbf{x}_i) \leq -\alpha ||\mathbf{x}_i - \boldsymbol{y}|| \leq f(\mathbf{x}_i) - f(\boldsymbol{y}) \leq \alpha ||\mathbf{x}_i - \boldsymbol{y}|| \leq \alpha C_b f(\mathbf{x}_i).$$

Then we have for all $1 \le i \le t$,

$$\frac{f(\boldsymbol{y})}{1+\alpha C_b} \le f(\mathbf{x}_i) \le \frac{f(\boldsymbol{y})}{1-\alpha C_b},$$

which implies that $||\mathbf{x}_i - \boldsymbol{y}|| \le \frac{C_b}{1-\alpha C_b} f(\boldsymbol{y})$. In other words, for all $1 \le i \le t$, $\mathbf{x}_i$ is inside the ball $B(\boldsymbol{y}, \frac{C_b}{1-\alpha C_b} f(\boldsymbol{y}))$.

As $B(\mathbf{x}_i, C_b f(\mathbf{x}_i))$ is bitten before the ball $B(\mathbf{x}_j, C_b f(\mathbf{x}_j))$, for all $1 \le i < j \le t$. We have for all $1 \le i < j \le t$, $||\mathbf{x}_i - \mathbf{x}_j|| \ge C_b f(\mathbf{x}_i)$. And $f()$ is α-Lipschitz implies that $f(\mathbf{x}_j) \le f(\mathbf{x}_i) + \alpha||\mathbf{x}_i - \mathbf{x}_j||$. Let $C_p = \frac{C_b}{2+\alpha C_b}$, we have

$$C_p f(\mathbf{x}_i) + C_p f(\mathbf{x}_j) \le 2C_p f(\mathbf{x}_i) + \alpha C_p ||\mathbf{x}_i - \mathbf{x}_j|| = ||\mathbf{x}_i - \mathbf{x}_j||.$$

In other words, for all $1 \le i < j \le t$, the interior of balls $B(\mathbf{x}_i, C_p f(\mathbf{x}_i))$ and $B(\mathbf{x}_j, C_p f(\mathbf{x}_j))$ do not overlap.

Notice that for all $1 \le i \le t$, the ball $B(\mathbf{x}_i, C_p f(\mathbf{x}_i))$ is inside the ball $B(\boldsymbol{y}, \frac{C_b}{1-\alpha C_b} f(\boldsymbol{y}) + C_p f(\mathbf{x}_i))$. As $f(\mathbf{x}_i) \le \frac{f(\boldsymbol{y})}{1-\alpha C_b}$, we know that $B(\boldsymbol{y}, \frac{C_b + C_p}{1-\alpha C_b} f(\boldsymbol{y}))$ also covers ball $B(\mathbf{x}_i, C_p f(\mathbf{x}_i))$ for all $1 \le i \le t$.

As the interior of balls $B(\mathbf{x}_i, C_p f(\mathbf{x}_i))$, $1 \le i \le t$ do not overlap, and $f(\mathbf{x}_i) \ge \frac{f(\boldsymbol{y})}{1+\alpha C_b}$, we have

$$\pi(\frac{C_b + C_p}{1-\alpha C_b} f(\boldsymbol{y}))^2 \ge \sum_{1 \le i \le t} \pi(C_p f(\mathbf{x}_i))^2 \ge t\pi(\frac{C_p}{1+\alpha C_b} f(\boldsymbol{y}))^2.$$

It implies that

$$t \le (\frac{(1+\alpha C_b)(C_b + C_p)}{C_p(1-\alpha C_b)})^2.$$

□

If the input is three dimensional domain, then we have the similar proof by using the volume argument. Also in order to have these theoretical guarantees make sense, C_b must be chosen such that $\alpha C_b < 1$.

Following from the above lemmas, we have the main theorem of our paper.

Theorem 0.3 (Good Point Set Guarantee) *Algorithm* Biting *generates a good point set for the meshless methods.*

Conclusion

A set of geometric criteria to define a good point set for the meshless methods is given. An algorithm, *Biting method*, is proposed to generate such a point set. Next step is to test the performance of our theoretically good algorithm, coupling it with a meshless method.

A Practical Variation. Maintaining an advancing front of arcs and computing the intersections of them can be time consuming. As a practical version of the algorithm, one can use squares or rectangles as the patches.

Adaptation. Adaptivity is a lot easier with the meshless methods, as one does not need to maintain the topology of the mesh. However, maintaining the quality of the point set is still an issue which can be achieved by maintaining a sphere packing. When the points are moved and bigger gaps are introduced, the domain can be oversampled and then the points, whose spheres overlap too much, can be eliminated.

Anisotropy. Anisotropy becomes important when there is a multiple length-scale phenomenon involved in the problems, such as shear banding strain localization or composite materials. Definition of a good point set given in the second section can easily modified using ellipse shaped patches to cover these types of problems. Point set generation of such problems can employ ellipse packing biting method [8].

Acknowledgments

The work reported here was supported, in part, by the Center for Process Simulation and Design (NSF DMS 98-73945) and the Center for Simulation of Advanced Rockets (DOE LLNL B341494). We would like to thank J.S. Chen of University of Iowa for his valuable comments on the criteria to define a good point set for meshless methods.

References

1. I. BABUSKA AND J.M. MELENK, *The Partition of Unity Finite Element Method* Technical Report BN-1185, Institute for Physical Science and Technology, University of Maryland, 1995.
2. T. BELYTSCHKO, Y.Y. LU, AND L. GU, *Element-free Galerkin Methods* International Journal for Numerical Methods in Engineering (37), 229-256, 1994.
3. T. BELYTSCHKO, Y. KRONGAUZ, D. ORGAN, M. FLEMING AND P. KRYSL, *Meshless Methods: An Overview and Recent Developments* Computer Methods in Applied Mechanics and Engineering (139), 3-47, 1996.
4. Y.J. CHOI AND S.J. KIM, *Node Generation Scheme for the MeshFree Method by Voronoi Diagram and Weighted Bubble Packing* Fifth U.S. National Congress on Computational Mechanics, Boulder, CO, 1999.
5. O. KLAAS AND M.S. SHEPHARD, *An Octree Based Partition of Unity Method for Three Dimensional Problems* Fifth U.S. National Congress on Computational Mechanics, Boulder, CO, 1999.
6. X.-Y.LI, S.-H.TENG AND A.ÜNGÖR, *Biting: Advancing Front meets sphere packing*, International Journal of Numerical Methods in Engineering, 1999, *to appear*
7. X.-Y. LI, S.-H. TENG AND A. ÜNGÖR, *Biting Spheres in 3D.* Proc. 8th Meshing Roundtable, 1999.
8. X.-Y. LI, S.-H. TENG AND A. ÜNGÖR, *Biting Ellipses to Generate Anisotropic Meshes.* Proc. 8th Meshing Roundtable, 1999.