Minimum-Energy Broadcasting in Static Ad Hoc Wireless Networks

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Abstract

Energy conservation is a critical issue in ad hoc wireless networks for node and network life since the nodes are powered by batteries only. One major approach for energy conservation is to route a communication session along the route which requires the lowest total energy consumption. This optimization problem is referred to as Minimum-Energy Routing. While the minimum-energy unicast routing problem can be solved in polynomial time by shortest-path algorithms, it remains open whether the minimum-energy broadcast routing problem can be solved in polynomial time despite the NP-hardness of its general graph version. Recently three greedy heuristics were proposed in [8]: MST (minimum spanning tree), SPT (shortest-path tree), and BIP (broadcasting incremental power). They evaluated their approaches through simulations [8], but little is known about their analytical performance. The main contribution of this paper is a *quantitative* characterization of their performances in terms of approximation ratios. By exploring geometric structures of Euclidean MSTs, we were able to prove that the approximation ratio of MST is between 6 and 12, and the approximation ratio of BIP is between $\frac{13}{3}$ and 12. On the other hand, we show that the approximation ratio of SPT is at least $\frac{n}{2}$, where n is the number of receiving nodes. To the best of our knowledge, these are the first analytical results for the minimum-energy broadcasting problem.

1 Introduction

Ad hoc wireless networks received significant attention in recent years due to their potential applications in battlefield, emergency disaster relief, and other application scenarios [7, 8]. Unlike wired networks or cellular networks, no wired backbone infrastructure is installed in ad hoc wireless networks. A communication session is achieved either through a single-hop transmission if the

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communication parties are close enough, or through relaying by intermediate nodes otherwise. Omnidirectional antennas are used by all nodes to transmit and receive signals. Such antennas are attractive due to their broadcast nature. A single transmission by a node can be received by many nodes within its vicinity. This feature is extremely useful for multicasting/broadcasting communications. For the purpose of energy conservation, each node can dynamically adjust its transmitting power based on the distance to the receiving node and the background noise. In the most common power-attenuation model [6], the signal power falls as $\frac{1}{r^\kappa}$ where r is the distance from the transmitter antenna and κ is a real *constant* between 2 and 4 dependent on the wireless environment.

Assume that all receivers have the same power threshold for signal detection, which is typically normalized to one. With these assumptions, the power required to support a link between two nodes separated by a distance r is r^κ. A key observation here is that relaying a signal between two nodes may result in lower total transmission power than communicating over a large distance due to the nonlinear power attenuation. As a simple illustration, consider three nodes $\mathbf{p}_1, \mathbf{p}_2$ and $\mathbf{p}_3$ with $\|\mathbf{p}_1\mathbf{p}_2\| > \|\mathbf{p}_1\mathbf{p}_3\|$ and assume $\kappa = 2$. See Figure 1. Node $\mathbf{p}_1$ wants to send a message to node $\mathbf{p}_2$. It has two options. It can transmit the signal directly to node $\mathbf{p}_2$, with a energy consumption of $\|\mathbf{p}_1\mathbf{p}_2\|^2$. Alternatively, it can send the message to node $\mathbf{p}_3$ and let it retransmit to node $\mathbf{p}_2$. The latter option has a total energy consumption of $\|\mathbf{p}_1\mathbf{p}_3\|^2 + \|\mathbf{p}_3\mathbf{p}_2\|^2$. Therefore if the angle $\angle\mathbf{p}_1\mathbf{p}_3\mathbf{p}_2$ is obtuse, the second option consumes less total energy. A crucial issue is then how to find a route with the minimum total energy consumption for a given communication session. This problem is referred to as *Minimum-Energy Routing* [7, 8].

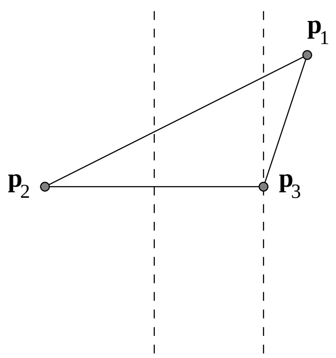

Figure 1: Reduce energy consumption through relaying.

Minimum-energy broadcast/multicast routing in a simple ad hoc networking environment was addressed by the pioneering work described in [8]. To assess the complexities *one at a time*, the nodes in the network are assumed to be static, namely without mobility, and randomly distributed in a two-dimensional plane. Nevertheless, as argued in [8], the impact of mobility can be incorporated into this static model because the transmitting power can be adjusted to accommodate the new locations of the nodes as necessary. In other words, the capability to adjust the transmission power provides considerable "elasticity" to the topological connectivity, and hence may reduce the need for hand-offs and tracking. In addition, as assumed in [8], there are sufficient bandwidth and transceiver resources. Under these assumptions, centralized (as opposed to distributed) algorithms were presented by [8] for minimum-energy broadcast/multicast routing. These centralized algorithms, in this simple networking environment, are expected to serve as the basis for further studies on distributed algorithms in a more practical network environment, where limited bandwidth and transceiver resources exist, as well as node mobility.

Three greedy heuristics were proposed in [8] for the minimum-energy broadcast routing problem: MST (minimum spanning tree), SPT (shortest-path tree), and BIP (broadcasting incremental power). They were evaluated through simulations in [8], but little is known about their analytical performance in terms of the approximation ratio. The approximation ratio of a heuristic, in this context, is the maximum ratio of the energy needed to broadcast a message based on the arborescence generated by this heuristic to the least necessary energy by any arborescence for any set of points. For the minimum-energy broadcast routing problem, one may come up with several seemingly reasonable greedy heuristics. Via simulation, it is difficult to determine which heuristic is better for an arbitrary configuration. Purely, for illustration, another slight variation of BIP, which is referred to as Broadcast Average Incremental Power (BAIP), is introduced in Section 3. Indeed, all the three heuristics proposed in [8] only have subtle differences. These subtle differences, however, can have a great impact on the analytical performance of these heuristics. In fact, we will show that the approximation ratios of MST and BIP are between 6 and 12 and between $\frac{13}{3}$ and 12 respectively; on the other hand, the approximation ratios of SPT and BAIP are at least $\frac{n}{2}$ and $\frac{4n}{\ln n} - o(1)$ respectively, where n is the number of nodes. To the best of our knowledge, these are the first *quantitative* characterizations of heuristics for the minimum-energy broadcast routing problem.

The remaining of this paper is organized as follows. In Section 2, we analyze the challenges for minimum-energy broadcast routing and briefly overview the three greedy heuristics developed in [8]. In Section 3, we construct some contrived instances to illustrate the poor performances of SPT and BAIP. These instances lead to the lower bounds on the approximation ratios of SPT and BAIP. In Section 4, we obtain lower bounds on the approximation ratios of MST and BIP by constructing some instances. In Section 5, we derive upper bounds on the approximation ratios of MST and BIP. A cornerstone to the analysis of the upper bounds is an elegant structure property of Euclidean MST which is explored in Section 6. Finally, in Section 7, we summarize our results and highlight several future research directions.

2 Preliminaries

We assume that the network nodes are given as a finite point[1] set P in a two-dimensional plane. For any real number κ, we use $G^{(\kappa)}$ to denote the weighted complete graph over P in which the weight of an edge e is $\|\mathbf{e}\|^{\kappa}$.

The minimum-energy unicast routing is essentially a shortest-path problem in $G^{(\kappa)}$. Consider any unicast path from a node $\mathbf{p} \in P$ to another node $\mathbf{q} \in P$:

$$\mathbf{p} = \mathbf{p}_0\mathbf{p}_1 \cdots \mathbf{p}_{m-1}\mathbf{p}_m = \mathbf{q}.$$

In this path, the transmission power of each node $\mathbf{p}_i$, $0 \leq i \leq m-1$, is $\|\mathbf{p}_i\mathbf{p}_{i+1}\|^{\kappa}$ and the transmission power of $\mathbf{p}_m$ is zero. Thus, the total transmission energy required by this path is

$$\sum_{i=0}^{m-1} \|\mathbf{p}_i\mathbf{p}_{i+1}\|^{\kappa},$$

which is the total weight of this path in G^{κ}. By applying any shortest-path algorithm such as Dijkstra's algorithm [2], one can solve the minimum-energy unicast routing problem.

However, for broadcast applications (in general multicast applications), Minimum-Energy Routing is far more challenging. Any broadcast routing is viewed as an arborescence (a directed tree) T, rooted at the source node of the broadcasting, that spans all nodes. We use $f_T(\mathbf{p})$ to denote the transmission power of the node $\mathbf{p}$ required by T. For any leaf node $\mathbf{p}$ of T, $f_T(\mathbf{p}) = 0$. For any internal node $\mathbf{p}$ of T,

$$f_T(\mathbf{p}) = \max_{\mathbf{pq} \in T} \|\mathbf{pq}\|^{\kappa},$$

in other words, the κ-th power of the longest distance between $\mathbf{p}$ and its children in T. The total energy required by T is $\sum_{\mathbf{p} \in P} f_T(\mathbf{p})$. Thus, the minimum-energy broadcast routing problem is different from the conventional link-based minimum spanning tree (MST) problem. Indeed, while the MST can be solved in polynomial time by algorithms such as Prim's algorithm and Kruskal's algorithm [2], it is still unknown whether the minimum-energy broadcast routing problem can be solved in polynomial time. In its general graph version, the minimum-energy broadcast routing can be shown to be NP-hard [3], and even worse, it can not be approximated within a factor of $(1-\epsilon)\log\Delta$, unless $NP \subseteq DTIME\left[n^{O(\log\log n)}\right]$, by an approximation-preserving reduction from the Connected Dominating Set problem [4], where Δ is the maximal degree and ϵ is any arbitrary small positive constant. However, this intractability of its general graph version does not necessarily imply the same hardness of its

[1] The terms node, point and vertex are interchangeable in this paper: node is a network term, point is a geometric term, and vertex is a graph term.

geometric version. In fact, as shown later in the paper, its geometric version can be approximated within a constant factor. Nevertheless, this suggests that the minimum-energy broadcast routing problem is considerably harder than the MST problem.

Three greedy heuristics were proposed for the minimum-energy broadcast routing problem in [8]. The MST heuristic first applies Prim's algorithm to obtain a MST, and then orients it as an arborescence rooted at the source node. The SPT heuristic applies Dijkstra's algorithm to obtain a SPT rooted at the source node. The BIP heuristic is the node version of Dijkstra's algorithm for SPT. It maintains, throughout its execution, a single arborescence rooted at the source node. The arborescence starts from the source node, and new nodes are added to the arborescence one at a time on the minimum incremental cost basis until all nodes are included in the arborescence. The incremental cost of adding a new node to the arborescence is the minimum additional power increased by some node in the current arborescence to reach this new node. The implementation of BIP is based on the standard Dijkstra's algorithm, with one fundamental difference on the operation whenever a new node q is added. Whereas Dijkstra's algorithm updates the node weights (representing the current knowing distances to the source node), BIP updates the cost of each link (representing the incremental power to reach the head node of the directed link). This update is performed by subtracting the cost of the added link pq from the cost of every link qr that starts from q to a node r not in the new arborescence.

The performance of these three greedy heuristics have been evaluated in [8] by simulation studies. However, their analytic performance in terms of the approximation ratio remains an open issue. In subsequent sections, we derive the bounds of their approximation ratios.

3 Greedy Is Not Always Good

Greedy approaches are the most natural and widely used techniques in designing practical heuristics for optimization problems. For the minimum-energy broadcast routing problem, one may think of many greedy heuristics, in addition to the three greedy heuristics proposed in [8]. The real challenge, however, is how to come up with a *provably good* one. Two greedy heuristics may only differ slightly, but this small variation can have a great impact on the analytical performance of these heuristics. In addition, some heuristics may perform quite well or even optimally in some situations, but may perform very poorly in some other situations. For the purpose of an illustration, in this section, we compare two example heuristics: one is SPT and the other is a new one. The "hard" instance constructed in this section can not only lead to lower bounds on the approximation ratios of these two heuristics, but also helps in designing an overall good greedy heuristic. For simplicity, we only consider $\kappa = 2$ in this section.

We begin with the SPT algorithm. Let ϵ be a sufficiently small positive number. Consider m nodes $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_m$ evenly distributed on a cycle of radius 1 centered at a node $\mathbf{o}$ (see Figure 2). For $1 \leq i \leq m$, let $\mathbf{q}_i$ be the point in the line segment $\mathbf{op}_i$ with $\|\mathbf{oq}_i\| = \epsilon$. We consider broadcasting from the node $\mathbf{o}$ to these $n = 2m$ nodes

$$\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_m, \mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_m.$$

The SPT is the superposition of paths $\mathbf{oq}_i\mathbf{p}_i$, $1 \leq i \leq m$. Its total energy consumption is

$$\epsilon^2 + m\,(1-\epsilon)^2.$$

On the other hand, if the transmission power of node $\mathbf{o}$ is set to 1, then the signal can reach all other points. Thus, the minimum energy consumed by all broadcasting methods is at most 1. So the approximation ratio of SPT is at least $\epsilon^2 + m\,(1-\epsilon)^2$. As $\epsilon \longrightarrow 0$, this ratio converges to $\frac{n}{2} = m$.

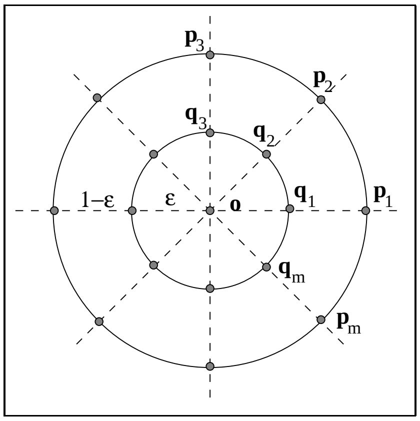

Figure 2: A bad instance for SPT.

The second greedy heuristic is similar to Chvatal's algorithm [1] for the Set Cover Problem and is a variation of BIP. Like BIP, an arborescence, which starts with the source node, is maintained throughout the execution of the algorithm. However, unlike BIP, many new nodes can be added one at a time. Similar to Chvatal's algorithm [1], the new nodes added are chosen to have the minimal *average* incremental cost, which is defined as the ratio of the minimum additional power increased by some node in the current arborescence to reach these new nodes to the number of these new nodes. We refer to this heuristic as the Broadcast Average Incremental Power, abbreviated by BAIP. In contrast to the $1+\log m$ approximation ratio of Chvatal's algorithm [1], where m is the largest set size in the Set Cover Problem, we show that the approximation ratio of BAIP is at least $\frac{4n}{\ln n} - o(1)$, where n is the number of receiving nodes.

Consider the following instance of minimum-energy broadcasting. All nodes lie on the x-axis with the source at the origin, the i-th receiving node at position $\sqrt{i}$ for $1 \leq i \leq n-1$, and the n-th receiving node at position $\sqrt{n-\epsilon}$ for some sufficiently small real number $\epsilon > 0$. For any $1 \leq k \leq n-1$, the minimal transmission power of the source to reach k receiving nodes is $\left(\sqrt{k}\right)^2 = k$, and thus the average incremental power cost at the origin to reach these k nodes is $\frac{k}{k} = 1$. On the other hand, the minimal transmission power of the source to reach all n receiving nodes is $\left(\sqrt{n-\epsilon}\right)^2 = n-\epsilon$, and the thus the average power efficiency is $\frac{n-\epsilon}{n} = 1 - \frac{\epsilon}{n}$. So BAIP will let the source to transmit at power $n-\epsilon$ to reach all nodes. However, the optimal routing is a directed path consisting of all nodes from left to right. So the minimum power consumption is

$$\begin{aligned}
&\sum_{i=1}^{n-1}\left(\sqrt{i}-\sqrt{i-1}\right)^2 + \left(\sqrt{n-\epsilon}-\sqrt{n-1}\right)^2. \\
&< \sum_{i=1}^{n}\left(\sqrt{i}-\sqrt{i-1}\right)^2 \\
&= 1 + \sum_{i=1}^{n-1}\frac{1}{\left(\sqrt{i+1}+\sqrt{i}\right)^2} \\
&\leq 1 + \sum_{i=1}^{n-1}\frac{1}{4i} \\
&\leq 1 + \frac{\ln(n-1)+1}{4} \\
&= \frac{\ln(n-1)+5}{4}.
\end{aligned}$$

Thus, the approximation ratio of BAIP is at least

$$\frac{n-\epsilon}{\sum_{i=1}^{n-1}\left(\sqrt{i}-\sqrt{i-1}\right)^2 + \left(\sqrt{n-\epsilon}-\sqrt{n-1}\right)^2}.$$

As $\epsilon \longrightarrow 0$, this ratio converges to

$$
\begin{aligned}
&\frac{n}{\sum_{i=1}^{n}\left(\sqrt{i}-\sqrt{i-1}\right)^{2}} \\
&=\frac{n}{1+\sum_{i=1}^{n-1}\frac{1}{\left(\sqrt{i+1}+\sqrt{i}\right)^{2}}} \\
&\geq\frac{n}{1+\sum_{i=1}^{n-1}\frac{1}{4i}} \\
&\geq\frac{n}{1+\frac{\ln(n-1)+1}{4}} \\
&=\frac{4n}{\ln(n-1)+5} \\
&=\frac{4n}{\ln n}-o(1).
\end{aligned}
$$

Interestingly, SPT generates the optimal solution in the second instance while BAIP can provide near-optimal or optimal solution for the first instance. On the other hand, MST and BIP have many similarities to SPT and BAIP but have constant approximation ratios as proved later. Thus, one must carefully design and select greedy heuristics.

4 Lower Bounds on Approximation Ratios

In this section, we derive lower bounds on approximation ratios of MST and BIP. We begin with MST.

Theorem 1 *The approximation ratio of MST is at least* 6 *for any* $\kappa \geq 2$.

Proof. Let ϵ be a sufficiently small positive real number. Consider seven nodes $\mathbf{o}, \mathbf{p}_1, \cdots, \mathbf{p}_6$ (see Figure 3), which satisfy

$$
\begin{aligned}
\|\mathbf{op}_1\| &= 1, \\
\|\mathbf{op}_i\| &= 1+\epsilon, 2 \leq i \leq 6; \\
\|\mathbf{p}_i\mathbf{p}_{i+1}\| &= 1, 1 \leq i \leq 5.
\end{aligned}
$$

Then for any $1 \leq i \leq 5$,

$$\angle \mathbf{p}_i\mathbf{op}_{i+1} < \frac{\pi}{3}$$

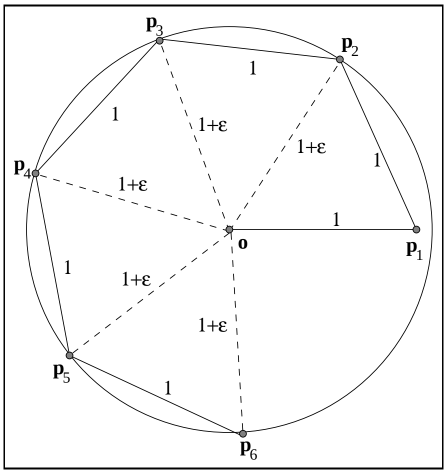

Figure 3: A bad instance for MST.

and

$$\angle \mathbf{p}_6 \mathbf{o} \mathbf{p}_1 > \frac{\pi}{3}.$$

Consider the two triangles $\mathbf{op_1p}_2$ and $\mathbf{op}_1\mathbf{p}_6$. Since

$$\|\mathbf{op}_2\| = \|\mathbf{op}_6\|$$

and

$$\angle \mathbf{p}_6 \mathbf{o} \mathbf{p}_1 > \angle \mathbf{p}_1 \mathbf{o} \mathbf{p}_2,$$

by Law of Cosine, we have

$$\|\mathbf{p}_1\mathbf{p}_6\| > \|\mathbf{p}_1\mathbf{p}_2\| = 1.$$

We consider the broadcasting from node $\mathbf{o}$ to nodes $\mathbf{p}_1, \cdots, \mathbf{p}_6$. Then the path $\mathbf{op}_1 \cdots \mathbf{p}_5\mathbf{p}_6$ is the unique MST. Its total energy consumption is 6. On the other hand, it is easy to show that the optimal routing is the star centered at node $\mathbf{o}$, whose total energy consumption is $(1+\epsilon)^\kappa$. Thus, the approximation ratio of MST is at least $\frac{6}{(1+\epsilon)^\kappa}$, which converges to 6 as $\epsilon \longrightarrow 0$. ■

Now we develop a lower bound on the approximation ratio of BIP.

Theorem 2 *The approximation ratio of BIP is at least $\frac{13}{3}$ for any $\kappa = 2$.*

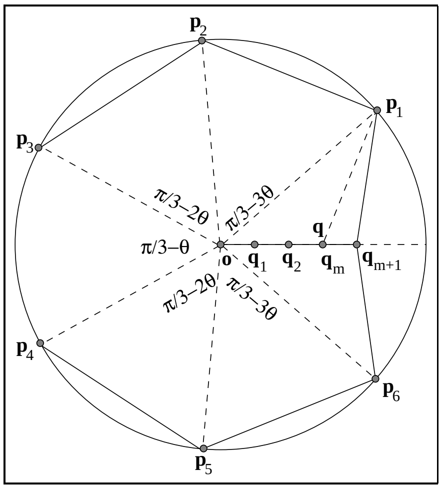

Figure 4: A instance for BIP.

Proof. Let θ be a sufficiently small positive real number. Consider six points $\mathbf{p}_1, \cdots, \mathbf{p}_6$ on a cycle of radius 1 centered at node $\mathbf{o}$ (see Figure 4), with

$$\begin{aligned}
\angle \mathbf{p}_1\mathbf{o}\mathbf{p}_2 &= \angle \mathbf{p}_5\mathbf{o}\mathbf{p}_6 = \frac{\pi}{3} - 3\theta, \\
\angle \mathbf{p}_2\mathbf{o}\mathbf{p}_3 &= \angle \mathbf{p}_4\mathbf{o}\mathbf{p}_5 = \frac{\pi}{3} - 2\theta, \\
\angle \mathbf{p}_3\mathbf{o}\mathbf{p}_4 &= \frac{\pi}{3} - \theta, \\
\angle \mathbf{p}_6\mathbf{o}\mathbf{p}_1 &= \frac{\pi}{3} + 11\theta.
\end{aligned}$$

Then

$$\begin{aligned}
&\|\mathbf{p}_1\mathbf{p}_2\| = \|\mathbf{p}_5\mathbf{p}_6\| < \\
&\|\mathbf{p}_2\mathbf{p}_3\| = \|\mathbf{p}_4\mathbf{p}_5\| < \\
&\|\mathbf{p}_3\mathbf{p}_4\| < 1 < \|\mathbf{p}_6\mathbf{p}_1\|.
\end{aligned}$$

Let $\mathbf{q}$ be the point in the perpendicular bisector of $\mathbf{p}_1\mathbf{p}_6$ such that $\mathbf{p}_1\mathbf{q}$ is perpendicular to $\mathbf{p}_1\mathbf{p}_2$. Choose a sufficiently large integer m such that

$$1 - \left(\frac{\|\mathbf{o}\mathbf{q}\|}{m}\right)^2 > \|\mathbf{p}_3\mathbf{p}_4\|^2.$$

Let $\mathbf{q}_1, \cdots, \mathbf{q}_{m+1}$ be the $m+1$ points on the ray $\mathbf{o}\mathbf{q}$ with

$$\|\mathbf{o}\mathbf{q}_i\| = \frac{i}{m}\|\mathbf{o}\mathbf{q}\|$$

for $1 \le i \le m+1$. Then $\mathbf{q}_m = \mathbf{q}$.

We consider broadcasting from point $\mathbf{o}$ to points $\mathbf{q}_1, \cdots, \mathbf{q}_{m+1}, \mathbf{p}_1, \cdots, \mathbf{p}_6$. The optimal solution is that the node $\mathbf{o}$ transmits at power 1 to reach all nodes. Now examine the output of the BIP algorithm. As m is sufficiently large, in the first $m+1$ steps, the points $\mathbf{q}_1, \cdots, \mathbf{q}_{m+1}$ are sequentially added, and the transmission power of the nodes $\mathbf{o}, \mathbf{q}_1, \cdots, \mathbf{q}_m$ all has the transmission power $\left(\frac{\|\mathbf{oq}\|}{m}\right)^2$. Since the angles

$$\angle \mathbf{p}_1 \mathbf{q}_{m+1} \mathbf{q}_m = \angle \mathbf{p}_6 \mathbf{q}_{m+1} \mathbf{q}_m > \frac{\pi}{2},$$

in the next two steps, the points $\mathbf{p}_1$ and $\mathbf{p}_6$ are added, and the transmission power of point $\mathbf{q}_{m+1}$ is $\|\mathbf{p}_1 \mathbf{q}_{m+1}\|^2$. At this moment, the incremental power of all points $\mathbf{o}, \mathbf{q}_1, \cdots, \mathbf{q}_m$ to reach any node $\mathbf{p}_i$ for $2 \leq i \leq 5$ is at least

$$1 - \left(\frac{\|\mathbf{oq}\|}{m}\right)^2 > \|\mathbf{p}_3 \mathbf{p}_4\|^2 > \|\mathbf{p}_1 \mathbf{p}_2\|^2 = \|\mathbf{p}_5 \mathbf{p}_6\|^2,$$

and the incremental power of point $\mathbf{q}_{m+1}$ to reach any node $\mathbf{p}_i$ for $2 \leq i \leq 5$ is also greater than $\|\mathbf{p}_1 \mathbf{p}_2\|^2 = \|\mathbf{p}_5 \mathbf{p}_6\|^2$ as

$$\angle \mathbf{p}_2 \mathbf{p}_1 \mathbf{q}_{m+1} = \angle \mathbf{p}_5 \mathbf{p}_6 \mathbf{q}_{m+1} > \angle \mathbf{p}_2 \mathbf{p}_1 \mathbf{q}_m = \frac{\pi}{2}.$$

Thus, in the subsequent two steps, the points $\mathbf{p}_2$ and $\mathbf{p}_5$ are added, and the transmission power of points $\mathbf{p}_1$ and $\mathbf{p}_6$ is $\|\mathbf{p}_1 \mathbf{p}_2\|^2 = \|\mathbf{p}_5 \mathbf{p}_6\|^2$. Similarly, in the last two steps, the points $\mathbf{p}_3$ and $\mathbf{p}_4$ are added, and the transmission power of points $\mathbf{p}_2$ and $\mathbf{p}_5$ is $\|\mathbf{p}_2 \mathbf{p}_3\|^2 = \|\mathbf{p}_4 \mathbf{p}_5\|^2$. The total power is

$$\begin{aligned} &(m+1)\left(\frac{\|\mathbf{oq}\|}{m}\right)^2 + \|\mathbf{p}_1 \mathbf{q}_{m+1}\|^2 + 2\|\mathbf{p}_1 \mathbf{p}_2\|^2 + 2\|\mathbf{p}_2 \mathbf{p}_3\|^2 \\ &= \frac{m+1}{m^2}\|\mathbf{oq}\|^2 + \|\mathbf{p}_1 \mathbf{q}_{m+1}\|^2 + 2\|\mathbf{p}_1 \mathbf{p}_2\|^2 + 2\|\mathbf{p}_2 \mathbf{p}_3\|^2. \end{aligned}$$

As $\theta \longrightarrow 0$ and $m \longrightarrow \infty$, the polygon $\mathbf{p}_1 \mathbf{p}_2 \mathbf{p}_3 \mathbf{p}_4 \mathbf{p}_5 \mathbf{p}_6$ converges to a regular hexagon, and the nodes $\mathbf{q}$ and $\mathbf{q}_{m+1}$ converges to the center of the triangle $\mathbf{op}_1 \mathbf{p}_6$. Thus, the total power consumption converges to $\frac{1}{3} + 4 = \frac{13}{3}$. Consequently, the approximation ratio of BIP is at least $\frac{13}{3} \approx 4.33$. ■

5 Upper Bounds on Approximation Ratios

We have given some lower bounds on the approximation ratios of MST and BIP by studying some special instances. However, upper bounds on the approximation ratios of these heuristics need to be analyzed for all possible instances. Our derivation of the upper bounds relies extensively on the geometric structures of Euclidean MSTs. We first observe that as long as the cost of a link is an increasing function of the Euclidean length of the link, the set of MSTs of any point set *coincides* with the set of Euclidean MSTs of the same point set. In fact, this can be followed from Prim's algorithm. In particular, for any spanning tree T of a finite point set P, parameter $\sum_{e\in T}\|e\|^2$ achieves its minimum if and only if T is an Euclidean MST of P. For any finite point set P, we use $mst\,(P)$ to denote an arbitrary Euclidean MST of P. The *radius* of a point set P is defined as

$$\inf_{\mathbf{p}\in P}\sup_{\mathbf{q}\in P}\|\mathbf{pq}\|.$$

Thus, a point set of radius one can be covered by a disk of radius one. A key result in this section is an upper bound on the parameter $\sum_{e\in mst(P)}\|e\|^2$ for any finite point set P of radius one. Note that the supreme of the total edge lengths of $mst\,(P)$, $\sum_{e\in mst(P)}\|e\|$, over all point sets P of radius one is infinity. Amazingly, however, the parameter $\sum_{e\in mst(P)}\|e\|^2$ is bounded from above by a constant for any point set P of radius one, as shown later. We use c to denote the supreme of $\sum_{e\in mst(P)}\|e\|^2$ over all point sets P of radius one. The next key theorem states that c is at most 12.

Theorem 3 *$6\le c\le 12$.*

The proof of this theorem involves complicated geometric arguments, and therefore we postpone it until Section 6. Note that for any point set P of radius one, the length of each edge in $mst\,(P)$ is at most one. Therefore, Theorem 3 implies that for any point set P of radius one and any real number $\kappa\ge 2$,

$$\sum_{e\in mst(P)}\|e\|^\kappa\le\sum_{e\in mst(P)}\|e\|^2\le c\le 12.$$

In the next, we explore a relation between the minimum energy required by a broadcasting and the energy required by the Euclidean MST of the corresponding point set.

Lemma 4 *For any point set P in the plane, the total anergy required by any broadcasting among P is at least $\frac{1}{c}\sum_{e\in mst(P)}\|e\|^\kappa$.*

Proof. Let T be an arborescence for broadcasting among P with the minimum energy consumption. For any non-leaf node $\mathbf{p}$ in T, let $T_{\mathbf{p}}$ be an Euclidean MST of the point set consisting $\mathbf{p}$ and all children of $\mathbf{p}$ in T. Suppose that the longest Euclidean distance between $\mathbf{p}$ and its children is r. Then, the transmission power of node $\mathbf{p}$ is r^κ, and all children of $\mathbf{p}$ lie in the disk centered at $\mathbf{p}$ with radius r. From the definition of c, we have

$$\sum_{e \in T_{\mathbf{p}}} \left(\frac{\|e\|}{r} \right)^\kappa \leq c,$$

which implies that

$$r^\kappa \geq \frac{1}{c} \sum_{e \in T_{\mathbf{p}}} \|e\|^\kappa .$$

Let T^* denote the spanning tree obtained by superposing of all $T_{\mathbf{p}}$'s for non-leaf nodes of T. Then, the total energy required by T is at least $\frac{1}{c} \sum_{e \in T^*} \|e\|^\kappa$, which is further no less than $\frac{1}{c} \sum_{e \in mst(P)} \|e\|^\kappa$. This completes the proof. ■

Consider any point set P in a two-dimensional plane. Let T be an arborescence oriented from some $mst(P)$. Then, the total energy required by T is at most $\sum_{e \in T_{\mathbf{p}}} \|e\|^\kappa$. From Lemma 4, this total energy is at most c times the optimum cost. Thus, the approximation ratio of the link-based MST heuristic is at most c. Together with Theorem 3, this observation leads to the following theorem.

Theorem 5 *The approximation ratio of the link-based MST heuristic is at most c, and therefore is at most 12.*

Finally, we derive an upper bound on the approximation ratio of the BIP heuristic. Once again, the Euclidean MST will play an important role.

Lemma 6 *For any broadcasting among a point set P in a two-dimensional plane, the total energy required by the arborescence generated by the BIP algorithm is at most $\sum_{e \in mst(P)} \|e\|^\kappa$.*

Proof. Remember that $G^{(\kappa)}$ is the complete graph over the point set P in which the weight of an edge e is $\|e\|^\kappa$. Let T be the arborescence generated by the algorithm BIP. We construct another weighted graph H over the same point set P according to the execution of BIP for generating T. Suppose that, during the execution of BIP, the nodes are added in the order $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n$, where $\mathbf{p}_1$ is the source node. Let T_i be the arborescence just after the node $\mathbf{p}_i$ is added. In H, the weight of edge $\mathbf{p}_i\mathbf{p}_{i+1}$ is equal to the *incremental* energy of the link from a node in T_i to $\mathbf{p}_{i+1}$ chosen during the execution of SPF; and the weight of any other edge, with at least one node not in T_i, is the same as that in $G^{(\kappa)}$. Note that for each edge $\mathbf{p}_i\mathbf{p}_{i+1}$, its weight in H is not more than its weight in $G^{(\kappa)}$. Therefore, for any spanning tree, its weight in H is no more than its weight in $G^{(\kappa)}$. On the other hand, the execution of Prim's algorithm on H will emulate the algorithm BIP on $G^{(\kappa)}$ in the sense that it will add the required nodes in the same order, and will output the path $\mathbf{p}_1\mathbf{p}_2\cdots\mathbf{p}_n$. The weight of this path in H is *exactly* the total energy required by T, but is at most the weight of any MST in $G^{(\kappa)}$. This implies that the total energy required by T is at most $\sum_{e\in mst(P)} \|e\|^{\kappa}$. This completes the proof. ■

From the above lemma and Lemma 4, we have the following result for the BIP algorithm similar to Theorem 5.

Theorem 7 *The approximation ratio of the BIP heuristic is at most c, and therefore is at most 12.*

6 Proof of Theorem 3

This section is devoted to the proof of Theorem 3. The lower bound is trivial as it can follow from the following instance consisting of seven points: the center of a regular hexagon and its six vertices. However, the deriving of the upper bound is challenging. We first introduce some geometric structures and notations to be used in this section. All angles are measured in radians and take values in the range $[0, \pi]$. For any three points $\mathbf{p}_1, \mathbf{p}_2$ and $\mathbf{p}_3$, the angle between the two rays $\mathbf{p}_1\mathbf{p}_2$ and $\mathbf{p}_1\mathbf{p}_3$ is denoted by $\angle\mathbf{p}_2\mathbf{p}_1\mathbf{p}_3$ or $\angle\mathbf{p}_3\mathbf{p}_1\mathbf{p}_2$. The *closed* infinite area inside the angle $\angle\mathbf{p}_2\mathbf{p}_1\mathbf{p}_3$, also referred to as a sector, is denoted by $\measuredangle\mathbf{p}_2\mathbf{p}_1\mathbf{p}_3$. The triangle determined by $\mathbf{p}_1, \mathbf{p}_2$ and $\mathbf{p}_3$ is denoted by $\triangle\mathbf{p}_2\mathbf{p}_1\mathbf{p}_3$. The *open* disk centered at $\mathbf{p}$ with radius r, denoted by $\mathbf{B}(\mathbf{p}, r)$, is the set of points such that every point has distance less than r from $\mathbf{p}$. The *lune* through points $\mathbf{p}_1$ and $\mathbf{p}_2$, denoted by $L(\mathbf{p}_1\mathbf{p}_2)$, is the intersection of the two open disks of radius $\|\mathbf{p}_1\mathbf{p}_2\|$ centered at $\mathbf{p}_1$ and $\mathbf{p}_2$ respectively (see Figure 5(a)). Thus, it consists of points whose distances from $\mathbf{p}_1$ and $\mathbf{p}_2$ are less than $\|\mathbf{p}_1\mathbf{p}_2\|$. The *open* diamond subtended by a line segment $\mathbf{p}_1\mathbf{p}_2$, denoted by $\mathbf{D}(\mathbf{p}_1\mathbf{p}_2)$, is the rhombus with sides each of which has length $\frac{\sqrt{3}}{3}\|\mathbf{p}_1\mathbf{p}_2\|$ (see Figure 5(b)). Note that the interior angles at $\mathbf{p}_1$ and $\mathbf{p}_2$ within $\mathbf{D}(\mathbf{p}_1\mathbf{p}_2)$ are equal to $\frac{\pi}{3}$.

The Euclidean MSTs have many nice structure properties [5]. Some basic properties are listed blow.

- Any pair of edges do not cross each other.
- The angles between any two edges incident to a common vertex is at least $\frac{\pi}{3}$.

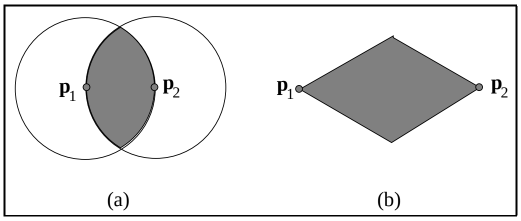

Figure 5: Illustration of (a) lune and (b) diamond.

- The length of each edge is at most the radius of the vertex set.
- The lune determined by each edge does not contain any other vertices.
- Let $\mathbf{p}_1\mathbf{p}_2$ be any edge. Then, the two endpoints of any other edge are either both outside $B(\mathbf{p}_1, \|\mathbf{p}_1\mathbf{p}_2\|)$, or both outside $B(\mathbf{p}_2, \|\mathbf{p}_1\mathbf{p}_2\|)$.

In this section, we first prove another structure property of the Euclidean MSTs, which is very essential to bound the constant c: *The diamonds of any two edges are disjoint.* The proof of this property will make use of the following lemma.

Lemma 8 *Let $\mathbf{p}_1, \mathbf{p}_2$ and $\mathbf{p}_3$ be any three points in the plane with $\angle\mathbf{p}_1\mathbf{p}_3\mathbf{p}_2 = \frac{2\pi}{3}$ and $\|\mathbf{p}_1\mathbf{p}_3\| = \|\mathbf{p}_2\mathbf{p}_3\|$ (see Figure 6). Let $\mathbf{p}_4$ be any point in $\measuredangle\mathbf{p}_1\mathbf{p}_3\mathbf{p}_2$ but outside $\triangle\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3$ with $\angle\mathbf{p}_2\mathbf{p}_1\mathbf{p}_4 = \alpha$. Then, $\mathbf{D}(\mathbf{p}_1\mathbf{p}_4) \subseteq \measuredangle\mathbf{p}_1\mathbf{p}_3\mathbf{p}_2$ if and only if either $\alpha \in \left[0, \frac{\pi}{3}\right)$ and $\|\mathbf{p}_1\mathbf{p}_4\| \leq \frac{\sin\frac{\pi}{3}}{\sin\left(\frac{\pi}{3}-\alpha\right)} \|\mathbf{p}_1\mathbf{p}_2\|$ or $\alpha \in \left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$.*

Proof. Note that $\mathbf{D}(\mathbf{p}_1\mathbf{p}_4) \subsetneq \measuredangle\mathbf{p}_1\mathbf{p}_3\mathbf{p}_2$ if $\alpha > \frac{2\pi}{3}$; and $\mathbf{D}(\mathbf{p}_1\mathbf{p}_4) \subseteq \measuredangle\mathbf{p}_1\mathbf{p}_3\mathbf{p}_2$ if $\alpha \in \left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$. So we now assume $\alpha \in \left[0, \frac{\pi}{3}\right)$. We fix α and calculate the maximum length of $\mathbf{p}_1\mathbf{p}_4$ such that $\mathbf{D}(\mathbf{p}_1\mathbf{p}_4) \subseteq \measuredangle\mathbf{p}_1\mathbf{p}_3\mathbf{p}_2$. This happens when $\mathbf{D}(\mathbf{p}_1\mathbf{p}_4)$ touches the ray $\mathbf{p}_3\mathbf{p}_2$, say at $\mathbf{x}$. We consider this extreme scenario. In this case,

$$\angle\mathbf{p}_3\mathbf{p}_1\mathbf{x} = \alpha, \angle\mathbf{p}_1\mathbf{x}\mathbf{p}_3 = \frac{\pi}{3} - \alpha.$$

Applying the Laws of Sine in $\triangle\mathbf{p}_1\mathbf{p}_3\mathbf{x}$, we have

$$\frac{\|\mathbf{p}_1\mathbf{x}\|}{\|\mathbf{p}_1\mathbf{p}_3\|} = \frac{\sin\frac{\pi}{3}}{\sin\left(\frac{\pi}{3} - \alpha\right)}.$$

On the other hand, as $\triangle\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3$ and $\triangle\mathbf{p}_1\mathbf{p}_4\mathbf{x}$ are similar,

$$\frac{\|\mathbf{p}_1\mathbf{p}_4\|}{\|\mathbf{p}_1\mathbf{p}_2\|} = \frac{\|\mathbf{p}_1\mathbf{x}\|}{\|\mathbf{p}_1\mathbf{p}_3\|} = \frac{\sin\frac{\pi}{3}}{\sin\left(\frac{\pi}{3} - \alpha\right)}.$$

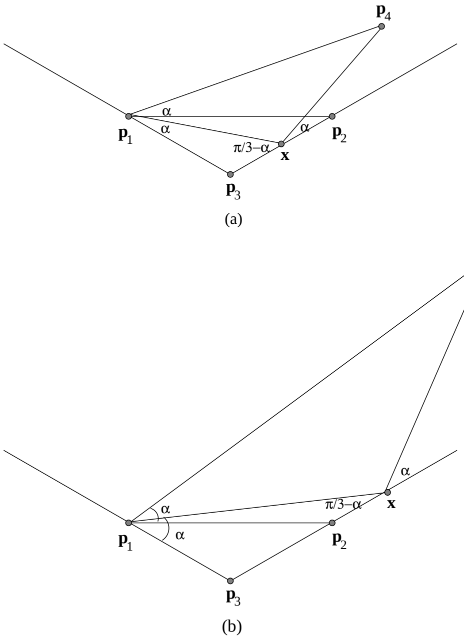

Figure 6: Two extreme cases for $\mathbf{D}(\mathbf{p}_1\mathbf{p}_4) \subseteq \measuredangle \mathbf{p}_1\mathbf{p}_3\mathbf{p}_2$.

Therefore, $\mathbf{D}(\mathbf{p}_1\mathbf{p}_4) \subseteq \measuredangle \mathbf{p}_1\mathbf{p}_3\mathbf{p}_2$ as long as $\|\mathbf{p}_1\mathbf{p}_4\| \leq \frac{\sin\frac{\pi}{3}}{\sin\left(\frac{\pi}{3}-\alpha\right)} \|\mathbf{p}_1\mathbf{p}_2\|$. ■

Next, we apply the above lemma to show that the diamond determined by any edge in an Euclidean MST is contained in some sector defined in the next lemma.

Lemma 9 *Let* $\mathbf{p}_1, \mathbf{p}_2$ *and* $\mathbf{p}_3$ *be any three points in the plane with* $\mathbf{p}_3$ *being outside* $L(\mathbf{p}_1\mathbf{p}_2)$. *Let* $\mathbf{p}_1'$ *(*$\mathbf{p}_2'$ *respectively) be the vertex of* $\mathbf{D}(\mathbf{p}_1\mathbf{p}_3)$ *(*$\mathbf{D}(\mathbf{p}_2\mathbf{p}_3)$ *respectively) which lies on the opposite side of the line* $\mathbf{p}_1\mathbf{p}_3$ *(*$\mathbf{p}_2\mathbf{p}_3$ *respectively) from* $\mathbf{p}_2$ *(*$\mathbf{p}_1$ *respectively) (see Figure 7). Then,* $\mathbf{D}(\mathbf{p}_1\mathbf{p}_2) \subseteq \measuredangle \mathbf{p}_1'\mathbf{p}_3\mathbf{p}_2'$.

Proof. We assume by symmetry that $\mathbf{p}_3$ is above the line $\mathbf{p}_1\mathbf{p}_2$ and to the right of the perpendicular bisector of $\mathbf{p}_1\mathbf{p}_2$. Then, $\|\mathbf{p}_1\mathbf{p}_3\| \geq \|\mathbf{p}_2\mathbf{p}_3\|$. Since $\mathbf{p}_3$ is outside $L(\mathbf{p}_1\mathbf{p}_2)$, $\|\mathbf{p}_1\mathbf{p}_3\| \geq \|\mathbf{p}_1\mathbf{p}_2\|$ and $\angle \mathbf{p}_1\mathbf{p}_3\mathbf{p}_2 < \frac{\pi}{2}$. Therefore,

$$\begin{aligned}
\angle \mathbf{p}_1'\mathbf{p}_3\mathbf{p}_2' &< \frac{\pi}{2} + \frac{\pi}{6} + \frac{\pi}{6} = \frac{5\pi}{6}, \\
\angle \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3 &\geq \frac{\pi}{3}, \\
\angle \mathbf{p}_2\mathbf{p}_1\mathbf{p}_3 &< \frac{\pi}{2}.
\end{aligned}$$

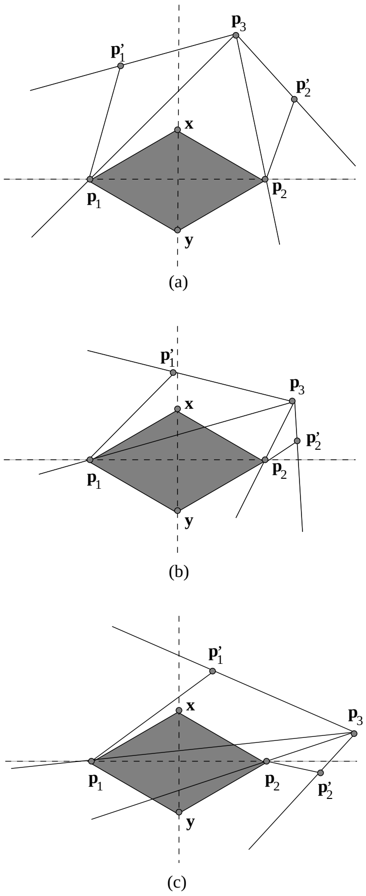

Figure 7: The three cases for Lemma 9

Let $\mathbf{x}$ and $\mathbf{y}$ be the other two vertices of $D(\mathbf{p}_1\mathbf{p}_2)$ which lie between the up side and the down side respectively of the line $\mathbf{p}_1\mathbf{p}_2$. It is sufficient to show that both $\mathbf{x}$ and $\mathbf{y}$ are within $\measuredangle\mathbf{p}'_1\mathbf{p}_3\mathbf{p}'_2$. This is true when $\angle\mathbf{p}_2\mathbf{p}_1\mathbf{p}_3 \geq \frac{\pi}{6}$ (see Figure 7(a)). So we assume that $\angle\mathbf{p}_2\mathbf{p}_1\mathbf{p}_3 < \frac{\pi}{6}$. In this case $\mathbf{x}$ is within $\triangle\mathbf{p}_1\mathbf{p}_3\mathbf{p}'_1$, and thus within $\measuredangle\mathbf{p}'_1\mathbf{p}_3\mathbf{p}'_2$, from Lemma 8 and $\|\mathbf{p}_1\mathbf{p}_3\| \geq \|\mathbf{p}_1\mathbf{p}_2\|$. If $\angle\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3 \leq \frac{5\pi}{6}$, then $\mathbf{y}$ is within $\measuredangle\mathbf{p}'_1\mathbf{p}_3\mathbf{p}_2 \subseteq \measuredangle\mathbf{p}'_1\mathbf{p}_3\mathbf{p}'_2$ (see Figure 7(b)). If $\angle\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3 > \frac{5\pi}{6}$, then

$$\begin{aligned}\angle\mathbf{p}_3\mathbf{p}_2\mathbf{y} &= \mathbf{2}\pi - \angle\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3 - \angle\mathbf{p}_1\mathbf{p}_2\mathbf{y} \\ &\geq \mathbf{2}\pi - \pi - \frac{\pi}{6} = \frac{5\pi}{6}\end{aligned}$$

which implies that the ray $\mathbf{p}_2\mathbf{y}$ does not intersect with the ray $\mathbf{p}_3\mathbf{p}'_2$ (see Figure 7(c)). So $\mathbf{y}$ is within $\measuredangle\mathbf{p}'_1\mathbf{p}_3\mathbf{p}'_2$. Therefore, in either case both $\mathbf{x}$ and $\mathbf{y}$ are within $\measuredangle\mathbf{p}'_1\mathbf{p}_3\mathbf{p}'_2$. This completes the proof. ■

Now we are ready to prove the "disjoint diamonds" property of Euclidean MSTs.

Lemma 10 *In any Euclidean MST, the two diamonds determined by any two edges are disjoint.*

Proof. The lemma is true when two edges are incident to a common vertex as the angle between them is at least $\frac{\pi}{3}$. So, we consider two edges $\mathbf{p}_1\mathbf{p}_2$ and $\mathbf{q}_1\mathbf{q}_2$ with distinct endpoints. We consider two cases.

Case 1: At least one of $\mathbf{p}_1\mathbf{p}_2$ and $\mathbf{q}_1\mathbf{q}_2$ does not cross the perpendicular bisector of the other. Without loss of generality, assume that $\mathbf{q}_1$ and $\mathbf{q}_2$ lie in the same side of the perpendicular bisector of $\mathbf{p}_1\mathbf{p}_2$ as $\mathbf{p}_1$ (see Figure 8(a)). Let $\mathbf{q}'_1$ ($\mathbf{q}'_2$ respectively) be the vertex of $\mathbf{D}(\mathbf{p}_1\mathbf{q}_1)$ ($\mathbf{D}(\mathbf{p}_1\mathbf{q}_2)$ respectively) which lies on the opposite side of the line $\mathbf{p}_1\mathbf{q}_1$ ($\mathbf{p}_1\mathbf{q}_2$ respectively) from $\mathbf{q}_2$ ($\mathbf{q}_1$ respectively). Then, from Lemma 9, $\mathbf{D}(\mathbf{q}_1\mathbf{q}_2) \subseteq \measuredangle\mathbf{q}'_1\mathbf{p}_1\mathbf{q}'_2$. On the other hand, since both $\mathbf{q}_1$ and $\mathbf{q}_2$ are outside $\mathbf{L}(\mathbf{p}_1\mathbf{p}_2)$, $\mathbf{D}(\mathbf{p}_1\mathbf{p}_2)$ is outside $\measuredangle\mathbf{q}'_1\mathbf{p}_1\mathbf{q}'_2$. Thus, $\mathbf{D}(\mathbf{p}_1\mathbf{p}_2)$ and $\mathbf{D}(\mathbf{q}_1\mathbf{q}_2)$ are disjoint.

Case 2: Both $\mathbf{p}_1\mathbf{p}_2$ and $\mathbf{q}_1\mathbf{q}_2$ cross the perpendicular bisector of the other. Without loss of generality, assume that $\mathbf{q}_1$ lies in the same side of the perpendicular bisector of $\mathbf{p}_1\mathbf{p}_2$ as $\mathbf{p}_1$ (see Figure 8(b)). Then, $\mathbf{p}_1$ must lie in the same side of the perpendicular bisector of $\mathbf{q}_1\mathbf{q}_2$ as $\mathbf{q}_1$, for otherwise

$$\|\mathbf{p}_2\mathbf{q}_1\| > \|\mathbf{p}_1\mathbf{q}_1\| > \|\mathbf{p}_1\mathbf{q}_2\| > \|\mathbf{p}_2\mathbf{q}_2\|,$$

i.e., both $\mathbf{p}_1$ and $\mathbf{p}_2$ lie in the same side of the perpendicular bisector of $\mathbf{q}_1\mathbf{q}_2$ as $\mathbf{q}_2$, which contradicts to the assumption. Since $\mathbf{q}_2$ is outside $\mathbf{L}(\mathbf{p}_1\mathbf{p}_2)$ and $\|\mathbf{p}_1\mathbf{q}_2\| > \|\mathbf{p}_2\mathbf{q}_2\|$, we have $\|\mathbf{p}_1\mathbf{q}_2\| > \|\mathbf{p}_1\mathbf{p}_2\|$. As $\|\mathbf{p}_1\mathbf{q}_2\| > \|\mathbf{p}_1\mathbf{q}_1\|$, $\mathbf{q}_2$ is outside $\triangle\mathbf{q}_1\mathbf{p}_1\mathbf{p}_2$. Similarly, any of these four points $\mathbf{p}_1, \mathbf{p}_2, \mathbf{q}_1$ and $\mathbf{q}_2$ is outside the triangle determined by the other three points. This implies that the convex

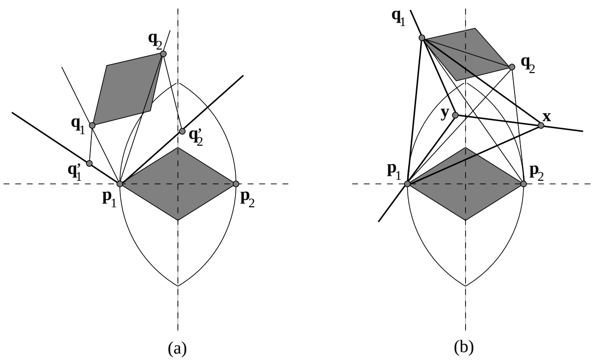

Figure 8: Two cases for Lemma 10.

hull determined by these four points is a quadrilateral. Note that $\mathbf{p}_1\mathbf{p}_2$ and $\mathbf{q}_1\mathbf{q}_2$ cannot be the two diagonals of the quadrilateral as they do not cross each other. Neither can be $\mathbf{p}_1\mathbf{q}_1$ and $\mathbf{p}_2\mathbf{q}_2$ as they are separated by the perpendicular bisector of $\mathbf{p}_1\mathbf{p}_2$. Thus, the two diagonals must be $\mathbf{p}_1\mathbf{q}_2$ and $\mathbf{p}_2\mathbf{q}_1$, and consequently the boundary of the quadrilateral is $\mathbf{p}_1\mathbf{p}_2\mathbf{q}_2\mathbf{q}_1$. From the previous argument its four sides are all less than its two diagonals, and hence its four inner angles are all more than $\frac{\pi}{3}$. Without loss of generality, we assume that $\|\mathbf{p}_1\mathbf{q}_1\| \geq \|\mathbf{p}_2\mathbf{q}_2\|$. Then $\|\mathbf{p}_1\mathbf{q}_1\| \geq \|\mathbf{p}_1\mathbf{p}_2\|$, for otherwise $\mathbf{q}_1$ would be inside $B(\mathbf{p}_1, \|\mathbf{p}_1\mathbf{p}_2\|)$ and $\mathbf{q}_2$ would be inside $B(\mathbf{p}_2, \|\mathbf{p}_1\mathbf{p}_2\|)$, which is impossible. Similarly, $\|\mathbf{p}_1\mathbf{q}_1\| \geq \|\mathbf{q}_1\mathbf{q}_2\|$. Therefore, both $\angle\mathbf{q}_1\mathbf{p}_1\mathbf{q}_2$ and $\angle\mathbf{p}_1\mathbf{q}_1\mathbf{p}_2$ are less than $\frac{\pi}{3}$. Since both $\angle\mathbf{q}_2\mathbf{p}_1\mathbf{p}_2$ and $\angle\mathbf{p}_2\mathbf{q}_1\mathbf{q}_2$ are less than $\frac{\pi}{2}$, we have

$$\angle\mathbf{q}_1\mathbf{p}_1\mathbf{p}_2, \angle\mathbf{p}_1\mathbf{q}_1\mathbf{q}_2 \in \left(\frac{\pi}{3}, \frac{5\pi}{6}\right).$$

Let $\mathbf{x}$ be the point inside $\measuredangle\mathbf{q}_1\mathbf{p}_1\mathbf{p}_2$ such that $\triangle\mathbf{p}_1\mathbf{q}_1\mathbf{x}$ is equilateral. Then, both $\mathbf{p}_1\mathbf{p}_2$ and $\mathbf{q}_1\mathbf{q}_2$ are outside $\triangle\mathbf{p}_1\mathbf{q}_1\mathbf{x}$. In addition,

$$\angle\mathbf{x}\mathbf{p}_1\mathbf{p}_2, \angle\mathbf{x}\mathbf{q}_1\mathbf{q}_2 \in \left(0, \frac{\pi}{2}\right)$$

and

$$\|\mathbf{p}_1\mathbf{x}\| \geq \|\mathbf{p}_1\mathbf{p}_2\|, \|\mathbf{q}_1\mathbf{x}\| \geq \|\mathbf{q}_1\mathbf{q}_2\|.$$

Let $\mathbf{y}$ be the center of $\triangle\mathbf{p}_1\mathbf{q}_1\mathbf{x}$. Then, from Lemma 8,

$$\mathbf{D}(\mathbf{p}_1\mathbf{p}_2) \subseteq \measuredangle\mathbf{p}_1\mathbf{y}\mathbf{x}, \mathbf{D}(\mathbf{q}_1\mathbf{q}_2) \subseteq \measuredangle\mathbf{q}_1\mathbf{y}\mathbf{x}.$$

This implies that $\mathbf{D}(\mathbf{p}_1\mathbf{p}_2)$ and $\mathbf{D}(\mathbf{q}_1\mathbf{q}_2)$ are disjoint.

■

Let P be any point set of radius one. According to Lemma 10, the total area covered by the diamonds through the edges in $mst\,(P)$ equals to

$$\frac{\sqrt{3}}{6}\sum_{e\in mst(P)}\|e\|^2.$$

Let $\mathbf{p}$ be any point in P. Then, every point in P has distance of at most one from $\mathbf{p}$. Since all edges of $mst(p)$ have lengths of at most one, all diamonds are contained in $B\left(\mathbf{p},\frac{2}{\sqrt{3}}\right)$. This implies that

$$\frac{\sqrt{3}}{6}\sum_{e\in mst(P)}\|e\|^2\leq\pi\left(\frac{2}{\sqrt{3}}\right)^2=\frac{4\pi}{3}.$$

Therefore

$$\sum_{e\in mst(P)}\|e\|^2\leq\frac{8\pi}{\sqrt{3}}\approx 14.51.$$

This estimation is quite loose and fails in getting the desired 12 upper bound. We now provide a tighter estimation which can lead to the 12 upper bound.

We observe that the total area of the diamonds is no more than the area of the disk $B\,(\mathbf{p},1)$ plus the sticking-out areas of these diamonds beyond $B\,(\mathbf{p},1)$. Let $\mathbf{D}\,(\mathbf{p}_1\mathbf{p}_2)$ be any diamond which sticks out $B\,(\mathbf{p},1)$, and let $\mathbf{q}$ be its vertex which is outside $B\,(\mathbf{p},1)$ (see Figure 9). Let $\mathbf{p}_1'$ ($\mathbf{p}_2'$ respectively) be the intersection between $\mathbf{p}_1\mathbf{q}$ ($\mathbf{p}_2\mathbf{q}$ respectively) and the boundary of $B\,(\mathbf{p},1)$. Then, the sticking-out area of $\mathbf{D}\,(\mathbf{p}_1\mathbf{p}_2)$ can be calculated by subtracting the area of the sector subtended by $\mathbf{pp}_1'$ and $\mathbf{pp}_2'$ from the area of the quadrilateral $\mathbf{pp}_1'\mathbf{qp}_2'$. The area of the quadrilateral $\mathbf{pp}_1'\mathbf{qp}_2'$ can be further calculated by summing up the areas of $\triangle\mathbf{pp}_1'\mathbf{p}_2'$ and $\triangle\mathbf{qp}_1'\mathbf{p}_2'$. As $\angle\mathbf{p}_1'\mathbf{qp}_2'$ is a constant $\frac{2\pi}{3}$, the area of $\triangle\mathbf{qp}_1'\mathbf{p}_2'$ is maximized when $\|\mathbf{qp}_1'\|=\|\mathbf{qp}_2'\|$. Let $\angle\mathbf{p}_1'\mathbf{pp}_2'=\alpha$, then $\alpha\in\left(0,\frac{\pi}{3}\right]$ and the sticking-out area of $\mathbf{D}\,(\mathbf{p}_1\mathbf{p}_2)$ is at most

$$S\,(\alpha)=\frac{1}{2}\sin\alpha+\frac{\sqrt{3}}{6}\,(1-\cos\alpha)-\frac{\alpha}{2}.$$

The area function $S\,(\alpha)$ has the following nice property.

Lemma 11 *For any $\alpha,\beta\in\left(0,\frac{\pi}{3}\right)$,*

1. *if $\alpha+\beta\leq\frac{\pi}{3}$, $S\,(\alpha)+S\,(\beta)\leq S\,(\alpha+\beta)$;*
2. *if $\alpha+\beta\geq\frac{\pi}{3}$, $S\,(\alpha)+S\,(\beta)\leq S\left(\alpha+\beta-\frac{\pi}{3}\right)+S\left(\frac{\pi}{3}\right)$.*

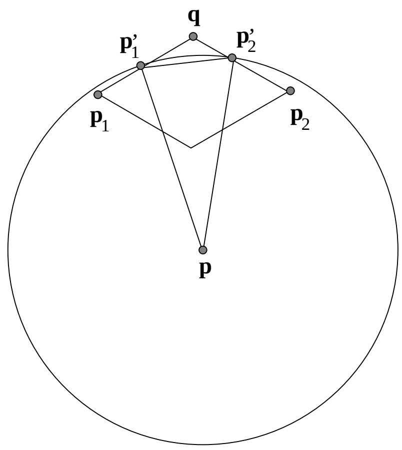

Figure 9: The calculation of the sticking-out area.

Proof. The lemma follows from the following two equalities: for any α and β,

$$\begin{aligned}
&S\left(\alpha+\beta\right)-S\left(\alpha\right)-S\left(\beta\right)\\
&=\frac{4\sqrt{3}}{3}\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\left(\frac{\pi}{6}-\frac{\alpha+\beta}{2}\right),\\
&S\left(\alpha+\beta-\frac{\pi}{3}\right)+S\left(\frac{\pi}{3}\right)-S\left(\alpha\right)-S\left(\beta\right)\\
&=\frac{4\sqrt{3}}{3}\sin\left(\frac{\alpha+\beta}{2}-\frac{\pi}{6}\right)\sin\left(\frac{\pi}{6}-\frac{\alpha}{2}\right)\sin\left(\frac{\pi}{6}-\frac{\beta}{2}\right).
\end{aligned}$$

We first prove the first equality.

$$
\begin{aligned}
& S(\alpha+\beta) - S(\alpha) - S(\beta) \\
& = \frac{1}{2}\left(\sin(\alpha+\beta) - \sin\alpha - \sin\beta\right) + \\
& \frac{\sqrt{3}}{6}\left((\cos\alpha + \cos\beta) - (\cos(\alpha+\beta) + 1)\right) \\
& = \left(\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha+\beta}{2} - \sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}\right) + \\
& \frac{\sqrt{3}}{3}\left(\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} - \cos^2\frac{\alpha+\beta}{2}\right) \\
& = \sin\frac{\alpha+\beta}{2}\left(\cos\frac{\alpha+\beta}{2} - \cos\frac{\alpha-\beta}{2}\right) + \\
& \frac{\sqrt{3}}{3}\cos\frac{\alpha+\beta}{2}\left(\cos\frac{\alpha-\beta}{2} - \cos\frac{\alpha+\beta}{2}\right) \\
& = \frac{2\sqrt{3}}{3}\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\left(\cos\frac{\alpha+\beta}{2} - \sqrt{3}\sin\frac{\alpha+\beta}{2}\right) \\
& = \frac{4\sqrt{3}}{3}\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\left(\frac{\pi}{6} - \frac{\alpha+\beta}{2}\right)
\end{aligned}
$$

Now we prove the second equality.

$$
\begin{aligned}
& S\left(\alpha+\beta-\frac{\pi}{3}\right) + S\left(\frac{\pi}{3}\right) - S(\alpha) - S(\beta) \\
& = \left(S(\alpha+\beta) - S(\alpha) - S(\beta)\right) - \\
& \left(S(\alpha+\beta) - S\left(\alpha+\beta-\frac{\pi}{3}\right) - S\left(\frac{\pi}{3}\right)\right) \\
& = \frac{4\sqrt{3}}{3}\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\left(\frac{\pi}{6} - \frac{\alpha+\beta}{2}\right) - \\
& \frac{4\sqrt{3}}{3}\sin\left(\frac{\alpha+\beta}{2} - \frac{\pi}{6}\right)\sin\frac{\pi}{6}\sin\left(\frac{\pi}{6} - \frac{\alpha+\beta}{2}\right) \\
& = \frac{4\sqrt{3}}{3}\sin\left(\frac{\alpha+\beta}{2} - \frac{\pi}{6}\right)\left(\sin\frac{\pi}{6}\sin\left(\frac{\alpha+\beta}{2} - \frac{\pi}{6}\right) - \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right) \\
& = \frac{2\sqrt{3}}{3}\sin\left(\frac{\alpha+\beta}{2} - \frac{\pi}{6}\right)\left(\cos\left(\frac{\pi}{3} - \frac{\alpha+\beta}{2}\right) - \cos\frac{\alpha-\beta}{2}\right) \\
& = \frac{4\sqrt{3}}{3}\sin\left(\frac{\alpha+\beta}{2} - \frac{\pi}{6}\right)\sin\left(\frac{\pi}{6} - \frac{\alpha}{2}\right)\sin\left(\frac{\pi}{6} - \frac{\beta}{2}\right).
\end{aligned}
$$

■

Suppose that there are k diamonds which stick out $B(\mathbf{p}, 1)$. For any $1 \leq i \leq k$, let α_i be the inner angle of the arc between the two intersection points of the boundary $B(\mathbf{p}, 1)$ and the boundary of the i-th sticking-out diamond. Then, $\alpha_i \in \left(0, \frac{\pi}{3}\right]$ and

$$\sum_{i=1}^{k} \alpha_i < 2\pi.$$

By repeatedly applying the two inequalities in Lemma 11, the total sticking-out area of the diamonds is

$$\begin{aligned}\sum_{i=1}^{k} S(\alpha_i) &\leq \left\lceil \frac{\sum_{i=1}^{k} \alpha_i}{\frac{\pi}{3}} \right\rceil S\left(\frac{\pi}{3}\right) \\ &\leq 6S\left(\frac{\pi}{3}\right) = 2\sqrt{3} - \pi.\end{aligned}$$

Thus, the total area of diamonds is at most

$$\pi + 2\sqrt{3} - \pi = 2\sqrt{3}.$$

Therefore,

$$\sum_{e \in mst(P)} \|e\|^2 \leq \frac{2\sqrt{3}}{\frac{\sqrt{3}}{6}} = 12.$$

This completes the proof Theorem 3.

7 Summary and Future Works

In this paper, we provided the theoretical performance analysis for the heuristics presented in [8]. The approximation ratio of SPT is at least $\frac{n}{2}$, and thus less favorable from the theoretical perspective. The other two heuristics, link-based MST and BIP, have constant-bounded approximation ratios. Specifically, the approximation ratio of the link-based MST heuristic is between 6 and c, which is at most 12; the approximation ratio of the BIP heuristic is between $\frac{13}{3}$ and $c \leq 12$. However, there are still several challenging issues for future research.

First of all, the computational complexity of the Minimum-Energy Broadcasting remains unknown. As mentioned in Section 1, the graph-version of this problem is at least as hard as the Connected Dominate Set problem. However, due to its geometric nature, this intractability of the graph version does not imply the same intractability of the geometric version. Indeed, while the Connected Dominating Set problem does not allow a constant-approximation ratio, the geometric version does on the contrary, for example, by MST or BIP.

Secondly, the exact value of the constant c remains unsolved. A tighter upper bound on c can lead to tighter upper bounds on the approximation ratios of both the link-based MST heuristic and the BIP heuristic. From the derivation of the 12 upper bound, we observe that there is still room to improve the upper bound. For example, it is very unlikely for the diamonds to fill the unit disk fully. At least this is true for small number of nodes. However, the treatment of large number of nodes is quite challenging, and more geometric properties of the Euclidean MSTs should be explored.

The third interesting problem is how to construct "harder" instances that can lead to better lower bounds on the approximation ratios of both the MST and BIP.

A major challenge, and a topic of continued research, is the development of distributed algorithms of MST and BIP. These algorithms should take advantage of the geometric properties for fast implementation. Furthermore, it is important to study the impact of limited bandwidth and transceiver resources, as well as to develop mechanisms to cope with node mobility [8].

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