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# Distributed Spanners with Bounded Degree for Wireless Ad Hoc Networks

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ABSTRACT

In this paper, we review some new distributed algorithms that construct sparse subgraphs with bounded degree of the unit disk graph efficiently for wireless ad hoc networks. They maintain a linear number of links while still preserving power-efficient routes for any pair of nodes. It was open whether the *Yao plus reverse Yao* graph and the *symmetric Yao* graph are spanners. We show that the Yao plus reverse Yao graph has a bounded power stretch factor 2 in civilized unit disk graph. In addition, we review a recent example by M. Grünewald *et al.* [6] to show that the *symmetric Yao* graph does not have a constant bounded stretch factor. Finally, we conduct simulations to study the practical performances of these structures. All structures have small power stretch factors for randomly generated unit disk graphs in our experiments.

*Keywords:* Wireless Ad Hoc Networks, Spanner, Topology Control, Power Consumption, Optimization.

## 1. Introduction

Wireless ad hoc networks have various applications in many areas and have drawn considerable attentions both from network engineers and theoretical researchers. One of the major concerns in designing wireless ad hoc networks is to save the power consumption as the wireless nodes are often powered by batteries only. In addition, the scalability is crucial for network operations as every node has limited resources such as memory. One effective approach to cope with these constraints is to maintain only a linear number of links using a *localized* construction method. Here a distributed algorithm constructing a graph $G$ is a *localized algorithm* if every node $u$ can exactly decide all edges incident on $u$ based only on the information of all nodes within a constant hops of $u$. At the same time, the sparseness property should not compromise too much on the power consumptions on communications. In this paper, we study how to construct a sparse spanner efficiently for a set of static or quasi-static wireless nodes such that, for any given pair of nodes, there is a power-efficient unicast route in the constructed network topology.

We consider a wireless ad hoc network consisting of a set $V$ of $n$ wireless nodes distributed in a two-dimensional plane. For the sake of simplicity, we assume that the nodes are static or static in a reasonable time period. In the most common power-attenuation model, the power needed to support a link $uv$ is $\|uv\|^\beta$, where $\|uv\|$ is the Euclidean distance between $u$ and $v$, and $\beta$ is a real constant between 2 and 4 dependent on the wireless transmission environment. By a proper scaling, we assume that all nodes have the maximum transmission ranges equal to one unit. These wireless nodes define a *unit disk graph* UDG(V) in which there is an edge between two nodes if and only if their Euclidean distance is at most one. The size of the unit disk graph could be as large as $O(n^2)$.

A trade-off can be made between the sparseness of the topology and the power efficiency. The power efficiency of any spanner is measured by its power stretch factor, which is defined as the maximum ratio of the minimum power needed to support the connection of two nodes in this spanner to the least necessary in the unit disk graph. Recently, Wattenhofer, *et al.* [19] proposed a two-phased power efficient network construction method using a variation of the Yao structure followed by a variation of the Gabriel graph. Li *et al.* [14] studied the power efficiency property of several well-known proximity graphs including the relative neighborhood graph, the Gabriel graph, and the Yao graph. These graphs are sparse and can be constructed locally and efficiently. They showed that the power stretch factor of the Gabriel graph is always one, and the power stretch factor of the Yao graph is bounded from above by a real constant while the power stretch factor of the relative neighborhood graph could be as large as $n-1$. Notice that all of these graphs do not have constant bounded node degrees, which can cause large overhead at some nodes in wireless networks. They further suggested to use another sparse topology, called *Yao and Sink* $\overrightarrow{YG}_k^*(V)$, which has both a constant bounded node degree and a constant bounded power stretch factor. We will review the algorithm to construct this topology in Section 3.

Li *et al.* [15] also defined another structure named *Yao plus reverse Yao*, denoted by $\overrightarrow{YY}_k(V)$ hereafter, which has a bounded node degree too. Their experiments showed that it has a small stretch factor in practice. However, they did not give a proof of its spanner property. In this paper, we review the algorithm constructing $\overrightarrow{YY}_k(V)$ and show that it is power-efficient in civilized graph.

Li *et al.* [13] and Stojmenovic [18] also considered another new undirected structure, called *symmetric Yao graph* $YS_k(V)$, which guarantees that the node degree is at most $k$. Stojmenovic [18] proved that it is a connected graph if UDG(V) is connected. Recently, Grünewald, *et al.* [6] showed that $YS_k(V)$ is not a spanner for length. We include their counter-example in Section 3.

The rest of the paper is organized as follows. In Section 2, we first give some definitions and review some results related to the distributed spanner for wireless networks. In Section 3, we review the methods to construct topologies $\overrightarrow{YG}^*_k(V)$, $\overrightarrow{YY}_k(V)$, and $YS_k(V)$ with bounded node degree. We give a simple proof that all structures are connected in Section 4. Our main result that $\overrightarrow{YY}_k(V)$ is a spanner in civilized graph is also presented in Section 4. Their practical performances are studied in Section 5. We conclude our paper in Section 6 by discussing some possible future works.

## 2. Preliminaries

In this section, we review some geometry definitions and notations that will be used later.

### *2.1. Spanner and Power Stretch Factor*

Constructing a spanner of a graph has been well studied by computational geometry community [1, 2, 4, 8, 16, 20]. Let $\Pi_G(u,v)$ be the shortest path connecting $u$ and $v$ in a graph $G$. Then, a graph $H$ is a spanner of $G$ if there exists a constant $t$ such that the length of $\Pi_H(u,v)$ is no more than $t$ times the length of $\Pi_G(u,v)$ for any two nodes $u$ and $v$. The constant $t$ is called the length stretch factor. Some researchers call it *dilation ratio*, or *spanning ratio*. However for wireless networks, we pay more attention on the power consumptions, the following definition of power stretch factor was introduced in [15].

Consider any unicast path $\Pi(u,v)$ in $G$ (could be directed) from a node $u \in V$ to another node $v \in V$:

$$\Pi(u,v) = v_0 v_1 \cdots v_{h-1} v_h,$$

where $u = v_0$, $v = v_h$ and $h$ is the number of hops of the path $\Pi$. The total *transmission power* $p(\Pi)$ consumed by this path $\Pi$ is defined as

$$p(\Pi) = \sum_{i=1}^{h} \|v_{i-1}v_i\|^\beta$$

Let $p_G(u,v)$ be the minimum power consumed by all paths connecting nodes $u$ and $v$ in $G$. The path in $G$ connecting $u, v$ and consuming the power $p_G(u,v)$ is called the *minimum-power path* in $G$ for $u$ and $v$.

Let $H$ be a subgraph of $G$. Its *power stretch factor* with respect to $G$ is then defined as

$$\rho_H(G) = \max_{u,v \in V} \frac{p_H(u,v)}{p_G(u,v)}$$

If $G$ is a unit disk graph, we use $\rho_H(V)$ instead of $\rho_H(G)$. For any $n$, let

$$\rho_H(n) = \sup_{|V|=n} \rho_H(V).$$

When the graph $H$ is clear from the context, it is dropped from notations. For the remainder of this section, we review some basic properties of the power stretch factor, which were studied and proved in [14, 15].

**Lemma 1** *For a constant $\delta$, $\rho_H(G) \leq \delta$ iff for any link $v_iv_j$ in graph $G$ but not in $H$, $p_H(v_i, v_j) \leq \delta \|v_iv_j\|^\beta$.*

**Lemma 2** *For any $H \subseteq G$ with a length stretch factor $\delta$, its power stretch factor is at most $\delta^\beta$ for any graph $G$.*

Lemma 1 implies that it is sufficient to analyze the power stretch factor of $H$ for each link in $G$ but not in $H$. ¿From Lemma 2, we know that a geometry structure $H$ with a constant length stretch factor $\delta$ has power stretch factor no more than $\delta^\beta$. However, the reverse is not true. Finally, the power stretch factor has the following monotonic property [14, 15].

**Lemma 3** *If $H_1 \subset H_2 \subset G$ then the power stretch factors of $H_1$ and $H_2$ satisfy $\rho_{H_1}(G) \geq \rho_{H_2}(G)$.*

### *2.2. Well-known Structures*

Various proximity subgraphs of the unit disk graph can be defined and be used in the topology control or other applications for wireless networks [3, 9, 12, 14, 17, 19].

The *relative neighborhood graph*, denoted by $RNG(V)$, consists of all edges $uv$ such that $\|uv\| \leq 1$ and there is no point $w \in lune(u, v)$. Here $lune(u, v)$ is the set of points $w$ such that $\|uw\| < \|uv\|$, and $\|wv\| < \|uv\|$. The *Gabriel graph*, denoted by $GG(V)$, consists of all edges $uv$ such that $\|uv\| \leq 1$ and the open disk using $uv$ as diameter does not contain any node from $V$. The *Yao graph* with an integer parameter $k \geq 6$, denoted by $\overrightarrow{YG}_k(V)$, is defined as follows. At each node $u$, any $k$ equally-separated rays originated at $u$ define $k$ cones. In each cone, choose the closest node $v$ to $u$ with distance at most one, if there is any, and add a directed link $\overrightarrow{uv}$. Ties are broken arbitrarily. Let $YG_k(V)$ be the undirected graph by ignoring the direction of each link in $\overrightarrow{YG}_k(V)$. See Figure 1 for an illustration.

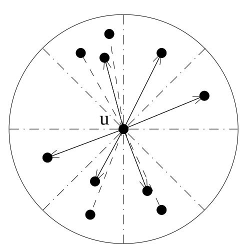


Figure 1: Neighbors of $u$ in Yao graph.

These graphs extend the conventional definitions of corresponding ones for the completed Euclidean graph; see [5, 10, 20]. It is well-known that $RNG(V)$ is a subgraph of $GG(V)$ and $YG_k(V)$ [2, 14, 15]. In addition, the following lemma can be proved [3, 2].

**Lemma 4** *If $UDG(V)$ is connected, $YG(V)$, $GG(V)$ and $RNG(V)$ contain Euclidean minimum spanning tree $EMST(V)$ as a subgraph.*

These graphs are sparse: $|RNG(V)| \leq 3n$, $|GG(V)| \leq 3n$, and $|\overrightarrow{YG}_k(V)| \leq kn$.

Bose, *et al.* [2] showed that the length stretch factor of $RNG(V)$ is $\Theta(n)$ and the length stretch factor of $GG(V)$ is $\Theta(\frac{4\pi\sqrt{2n-4}}{3})$. Several papers showed that the Yao graph $\overrightarrow{YG}_k(V)$ has length stretch factor at most $\frac{1}{1-2\sin\frac{\pi}{k}}$. Recently, Li, *et al.* [14] studied the power efficiency property of these well-known proximity graphs. They showed that the power stretch factor of $GG(V)$ is always 1, and the power stretch factor of the Yao graph is at most $\frac{1}{1-(2\sin\frac{\pi}{k})^\beta}$ while the power stretch factor of $RNG(V)$ could be as large as $n-1$. All these structures can be constructed locally.

### *2.3. Bounded Degree?*

The sparseness of these well-known proximity graphs implies that the average node degree is bounded by a constant. However, Li *et al.* [14] showed that the maximum node degree could be as large as $n-1$ as shown in Figure 2. The instance consists of $n-1$ points lying on the unit circle centered at a node $u \in V$. Then each edge $uv_i$ belongs to the $RNG(V)$, $GG(V)$ and $\overrightarrow{YG}_k(V)$. Thus, node $u$ has degree $n-1$ (in-degree for $\overrightarrow{YG}_k(V)$) in $RNG(V)$, $GG(V)$ and $\overrightarrow{YG}_k(V)$, although $\overrightarrow{YG}_k(V)$ has a bounded out-degree $k$ for each node.

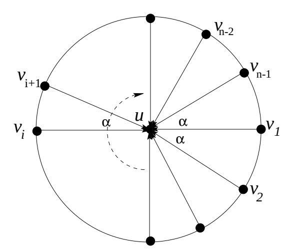


Figure 2: Node $u$ has degree (or in-degree) $n-1$.

Because wireless nodes have limited resources, we prefer the node degree be bounded by a constant. Unbounded degree (or in-degree) at node $u$ will often cause large overhead at $u$. On the other hand, bounded degree will also give us advantages when apply several routing algorithms. Therefore, it is often imperative to construct a sparse network topology such that both the in-degree and the out-degree are bounded by a constant while it is still power-efficient.

### *2.4. Civilized Graph*

When we prove the spanner property of one of the new structures, $\overrightarrow{YY}_k(V)$, we consider it in the *civilized unit disk graph* instead of the general unit disk graph. Here a $UDG(V)$ is a civilized graph if the distance between any two nodes in this graph is larger than a constant $\lambda$. In [7], they called the civilized unit disk graph as the *$\lambda$-precision unit disk graph.* Notice the wireless devices in wireless network can not be too close or overlapped, so it is reasonable to model the wireless ad hoc network as a civilized unit disk graph.

## 3. Previous Results

We first review the definitions, the constructions and the properties of some bounded degree structures proposed recently [6, 13, 14, 15, 18] All of them can be constructed locally and have bounded node degrees.

### *3.1. Construction of Bounded Degree Structures*

Assume that each node $v_i$ of $V$ has a unique identification number $ID(v_i) = i$. The identity of a directed link $\overrightarrow{uv}$ is defined as

$$ID(\overrightarrow{uv}) = (\|uv\|, ID(u), ID(v)).$$

Then we can order all directed links (at most $n(n-1)$ such links) in an increasing order of their identities. Here the identities of two links are ordered based on the following rules: $ID(\overrightarrow{uv}) > ID(\overrightarrow{pq})$ if

1. $\|uv\| > \|pq\|$ or
2. $\|uv\| = \|pq\|$ and $ID(u) > ID(p)$ or
3. $\|uv\| = \|pq\|$, $u = p$ and $ID(v) > ID(q)$.

Correspondingly, the rank, denoted by $rank(\overrightarrow{uv})$, of each directed link $\overrightarrow{uv}$ is its order in the sorted directed links. Notice that, we only have to consider the links with length no more than one.

The Yao graph $\overrightarrow{YG}_k(V)$ can be constructed as follows.

Algorithm: Constructing-YG

Each node $u$ divides the space by $k$ equal-sized cones centered at $u$. We generally assume that the cone is half open and half-close. Node $u$ chooses a node $v$ from each cone so the directed link $\overrightarrow{uv}$ has the smallest $ID(\overrightarrow{uv})$ among all directed links $\overrightarrow{uv}_i$ with $v_i$ in that cone, if there is any. Let $\overrightarrow{YG}_k(V)$ be the union of all chosen directed links.

#### 3.1.1. Yao and Sink

Arya, *et al.* [1] presented an innovative technique to generate a bounded degree graph with a constant length stretch factor. In [14, 15], the authors applied the same technique to construct a sparse network topology, *Yao and sink graph*, with a bounded degree and a bounded power stretch factor. The technique is to replace the directed star consisting of all links toward a node $u$ by a directed tree $T(u)$ with $u$ as the sink. Tree $T(u)$ is constructed recursively. The algorithm is as follows, (see [15] for more detail).

Algorithm: Constructing-YG*

1. First, construct the graph $\overrightarrow{YG}_k(V)$. Each node $u$ will have a set of in-coming nodes $I(u) = \{v \mid \overrightarrow{vu} \in \overrightarrow{YG}_k(V)\}$.

2. For each node $u$, use the following Tree($u$,$I(u)$) to build tree $T(u)$.

Algorithm: Constructing-$T(u)$ Tree($u$,$I(u)$)

1. To partition the unit disk centered at $u$, we choose $k$ equal-sized cones centered at $u$: $C_1(u)$, $C_2(u)$, $\cdots C_k(u)$.

2. Node $u$ finds the node $y_i \in I(u)$ in $C_i(u)$ with the smallest $ID(\overrightarrow{y_i u})$, for $1 \leq i \leq k$, if there is any. Link $\overrightarrow{y_i u}$ is added to $T(u)$ and $y_i$ is removed from $I(u)$. For each cone $C_i(u)$, if $I(u) \cap C_i(u)$ is not empty, call Tree($y_i$,$I(u) \cap C_i(u)$) and add the created edges to $T(u)$.

Notice that, node $u$ constructs the tree $T(u)$ and then broadcasts the structure of $T(u)$ to all nodes in $T(u)$. Figure 3 (a) illustrates a directed star centered at $u$ and Figure 3 (b) shows the directed tree $T(u)$ constructed to replace the star with $k = 8$. The union of all trees $T(u)$ is called the *sink structure* $\overrightarrow{YG}_k^*(V)$.

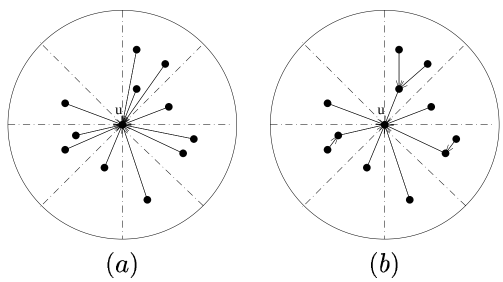


Figure 3: (a) Star formed by links toward to $u$. (b) Directed tree $T(u)$ sinked at $u$.

### 3.1.2. Yao plus reverse Yao

In [15], the authors defined another structure named *Yao plus reverse Yao*, denoted by $\overrightarrow{YY}_k(V)$. The construction algorithm is as follows.

Algorithm: Constructing-YY

1. First, construct the graph $\overrightarrow{YG}_k(V)$.

2. Node $u$ chooses a node $v$ from each cone, if there is any, so the directed link $\overrightarrow{vu}$ has the smallest $ID(\overrightarrow{vu})$ among all directed links computed in the first step in that cone.

The union of all chosen directed links in the second step is the final network topology, denoted by $\overrightarrow{YY}_k(V)$. If the link directions are ignored, the graph is denoted as $YY_k(V)$. Compared with $\overrightarrow{YG}_k^*(V)$, $\overrightarrow{YY}_k(V)$ replaces the directed tree $T(u)$ by a directed star (See Figure 4) consisting of at most k links toward a node $u$.

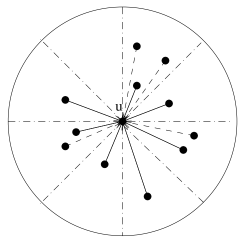


Figure 4: Node $u$ chooses the shortest directed link toward $u$ in each cone instead of constructing tree $T(u)$.

3.1.3. Symmetric Yao

In [13, 18], they considered another undirected structure, called *symmetric Yao graph* $YS_k(V)$, which guarantees that the node degree is at most $k$. Symmetric Yao graph can be constructed as follows.

Algorithm: Constructing-YS

1. First, construct the graph $\overrightarrow{YG}_k(V)$.
2. An edge $uv$ is selected to graph $YS_k(V)$ if and only if both directed edges $\overrightarrow{uv}$ and $\overrightarrow{vu}$ are in $\overrightarrow{YG}_k(V)$.

*3.2. Known Properties of Structures*

The algorithm which constructs Yao and sink structure uses a directed tree $T(u)$ to replace the directed star for each node $u$. Therefore, it does not change the connectivity of the structure. We already know that $\overrightarrow{YG}_k$ is strongly connected if UDG(V) is connected, so does $\overrightarrow{YG}_k^*$.

In [14], they proved the node degree of $\overrightarrow{YG}_k^*$ is bounded by $(k+1)^2-1$ (its in-degree is bounded by $k(k+1)$ while the maximum out-degree is $k$.) and its power stretch factor is at most $(\frac{1}{1-(2\sin\frac{\pi}{k})^\beta})^2$.

We comment it here that the sink structure and the Yao graph structure do not need to have the same number of cones, and the cones centered at different nodes do not need to be aligned. For setting up a power-efficient wireless networking, each node $u$ finds all its neighbors in $\overrightarrow{YG}_k(V)$, which can be done in linear time proportional to the number of nodes within its transmission range.

However, the construction of $\overrightarrow{YG}_k^*(V)$ is actually more complicated and the performance gain compared with $\overrightarrow{YY}_k(V)$ is not so obvious in practice as shown by our experimental results in Section 5.

For the Yao plus reverse Yao graph, it is obvious that both the out-degree and in-degree of a node in $\overrightarrow{YY}_k(V)$ are bounded by $k$. This implies that $\overrightarrow{YY}_k(V)$ is a sparse graph. ¿From the construction algorithm, we also know $\overrightarrow{YY}_k(V)$ is a subgraph of $\overrightarrow{YG}_k^*(V)$, because all the links selected by node $u$ in the second step are in the directed tree $T(u)$ built by node $u$ in $\overrightarrow{YG}_k^*(V)$.

In [15], by using induction they proved that $\overrightarrow{YY}_k(V)$ is strongly connected if $UDG(V)$ is connected. Moreover, their experiments showed that $\overrightarrow{YY}_k(V)$ has a small stretch factor in practice. However, they did not give a proof of its spanner property. In next section, we will show that it is power-efficient in civilized graph.

For the symmetric Yao graph $YS_k(V)$, it is obvious that its node degree is bounded by $k$. Recently Grünewald *et al.* [6] and Stojmenovic [18] both proved that $YS_k(V)$ is strongly connected if $UDG(V)$ is connected. But the proofs were long and complex. We will give a simpler proof in next section.

Grünewald *et al.* [6] showed that $YS_k(V)$ is not a spanner for length or power by giving a counter example, whose basic idea is similar to the counter example for RNG proposed by Bose *et al.* [2]. For the completeness of the presentation, we still review the counter example here.

Let nodes $v_1$ and $v_0$ have distance half unit from each other. Assume the $i$th cone of $v_1$ contains $v_0$, and the $i'$th cone of $v_0$ contains $v_1$. Then draw two lines $l_1 = v_1v_3$ and $l_2 = v_0v_2$ such that both the angles $\angle v_3v_1v_0$ and $\angle v_2v_0v_1$ are $\frac{\pi}{2} - \alpha$, where $\alpha$ is a very small positive number. We first consider even $n$, say $n = 2m$. Figure 5 illustrates the construction of the point set $V$. The node $v_{2j}$ is placed on $l_2$ in the $i$th cone of $v_{2j-1}$ and it is very close to the upper boundary of the $i$th cone of $v_{2j-1}$. The node $v_{2j+1}$ is placed on $l_1$ in the $i'$th cone of $v_{2j}$ close to the upper boundary of that cone. Using this method, we place all nodes from $v_2$ to $v_{2m}$ in order. Then it is easy to show that the $YS_k(V)$ does not contain any edge $v_{2j}v_{2j+1}$ and $v_{2j+1}v_{2j+2}$ for $0 \le j \le m-1$. The nearest neighbor of $v_{2j}$ is $v_{2j+1}$, but for $v_{2j+1}$, the nearest neighbor is $v_{2j+2}$. So although in $YS_k(V)$ there is a path from $v_1$ to $v_2$, its length is $\|v_1v_{2m-1}\| + \|v_{2m-1}v_{2m}\| + \|v_{2m}v_2\|$. So when $\alpha$ is appropriately small, the length stretch factor of $YS_k(V)$ cannot be bounded by a constant. Similarly, its power stretch factor cannot be bounded also. When $n$ is odd, the construction is similar.

Nevertheless, our experiments show that $YS_k(V)$ has a small power stretch factor in practice.

## 4. Our New Results

### *4.1. A simple proof for connectivity*

Grünewald *et al.* [6] and Stojmenovic [18] already proved the connectivity of $YS_k(V)$ by induction. When given any node $u$, there are two nodes that are equal distance to $u$, we give a simpler proof here.

**Theorem 1** *The graph $YS_k(V)$ is connected if $UDG(V)$ is connected and $k \ge 6$.*

PROOF. First we prove that $RNG(V) \subseteq YS_k(V)$. From the definition of $RNG(V)$, there is an edge $uv$ in $RNG(V)$ if and only if the shaded region shown in Figure 6 (b) is empty. If the shaded region shown in Figure 6 (b) is empty then the shaded region shown in Figure 6 (a) must be empty when $\theta \le \frac{\pi}{3}$, i.e. $k \ge 6$. So any edge $uv$ in $RNG(V)$ must be in $YS_k(V)$ also.

Remember that $RNG(V)$ is connected if $UDG(V)$ is connected from Lemma

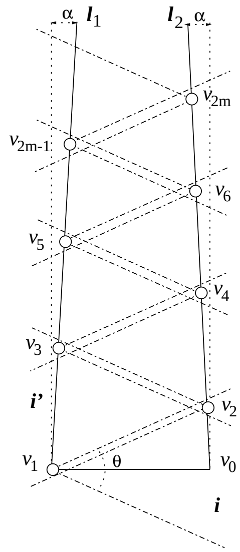


Figure 5: An example that $YS_k(V)$ has a large stretch factor.

4. So the symmetric Yao graph, which is a supergraph of $RNG(V)$, is strongly connected when $RNG(V)$ is connected. $\square$

By definitions, it is easy to show that, given a set of nodes in general position,

$$YS_k(V) \subseteq YY_k(V) \subseteq YG_k^*(V),\ and\ YY_k(V) \subseteq YG_k(V). \tag{1}$$

Then all such graphs are connected, if $UDG(V)$ is connected.

*4.2. $YY_k(V)$ is a spanner in civilized graph*

We now prove that $\overrightarrow{YY}_k(V)$ is a spanner in civilized graph. Remember that in a civilized graph the distance between any two nodes is at least $\lambda$.

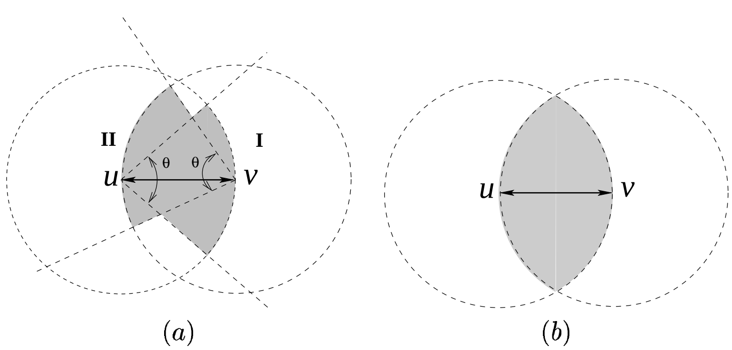


Figure 6: (a) if $uv$ is an edge of $YS_k(V)$, then the shaded region must be empty. (b) if $uv$ is an edge of $RNG(V)$, then the shaded region must be empty.

**Theorem 2** *The power stretch factor of the directed topology $\overrightarrow{YY}_k(V)$ is bounded by a constant $\rho$ in civilized graph.*

PROOF. We actually prove the following claims by induction on the rank of the directed links:

1. There is a constant $\delta > 1$, such that for any directed link $\overrightarrow{v_iv_j}$ in the graph $\overrightarrow{YG}_k(V)$, the minimum power consumption in $\overrightarrow{YY}_k(V)$ from $v_i$ to $v_j$ is no more than $\delta\|v_iv_j\|^\beta$. Specifically, we show that

$$p_{\overrightarrow{YY}_k(V)}(v_i, v_j) \leq \delta\|v_iv_j\|^\beta.$$

2. There is a constant $\rho > \delta$, such that for any directed link $\overrightarrow{v_iv_j}$ not in the graph $\overrightarrow{YG}_k(V)$, the minimum power consumption in $\overrightarrow{YY}_k(V)$ from $v_i$ to $v_j$ is no more than $\rho\|v_iv_j\|^\beta$. In other words, we show that

$$p_{\overrightarrow{YY}_k(V)}(v_i, v_j) \leq \rho\|v_iv_j\|^\beta.$$

The directed link with the rank one is obviously in $\overrightarrow{YY}_k(V)$, thus, the first claim holds. Assume that the claims are true for all links with rank at most $r$. Then consider the directed link $\overrightarrow{ut}$ with rank $r+1$.

*Case 1*: link $\overrightarrow{ut}$ does not belong to $\overrightarrow{YG}_k(V)$. Then there is a directed path $\Pi_{\overrightarrow{YG}_k}(u,t) = q_1q_2\cdots q_h$ from $u$ to $t$ in graph $\overrightarrow{YG}_k(V)$, where $q_1 = u$ and $q_h = t$. Let $v$ be node $q_2$. Then we have

$$rank(\overrightarrow{uv}) = (\|\overrightarrow{uv}\|, ID(u), ID(v)) < rank(\overrightarrow{ut}) = (\|\overrightarrow{ut}\|, ID(u), ID(t)) \qquad (2)$$

because of the selection method of the first step. Similarly,

$$rank(\overrightarrow{vt}) = (\|\overrightarrow{vt}\|, ID(v), ID(t)) < rank(\overrightarrow{ut}) = (\|\overrightarrow{ut}\|, ID(u), ID(t)) \qquad (3)$$

because $\|\overrightarrow{vt}\| < \|\overrightarrow{ut}\|$. Then we can apply the induction on $\overrightarrow{uv}$ and $\overrightarrow{vt}$. Notice that here $\overrightarrow{vt}$ may not belong to $\overrightarrow{YG}_k(V)$. Consequently, we have

$$p_{\overrightarrow{YY}_k(V)}(u,v) \leq \delta\|uv\|^\beta, p_{\overrightarrow{YY}_k(V)}(v,t) \leq \rho\|vt\|^\beta. \qquad (4)$$

Therefore,

$$p_{\overrightarrow{YY}_k(V)}(u,t) \leq p_{\overrightarrow{YY}_k(V)}(u,v) + p_{\overrightarrow{YY}_k(V)}(v,t) \leq \delta\|uv\|^\beta + \rho\|vt\|^\beta. \qquad (5)$$

There are two subcases here. Either $\angle uvt$ is acute or not.

*Subcase 1.1*: the angle $\angle uvt$ is not acute. Then we have

$$\|uv\|^\beta + \|vt\|^\beta \leq \|ut\|^\beta.$$

It implies that

$$\delta\|uv\|^\beta + \rho\|vt\|^\beta < \rho(\|uv\|^\beta + \|vt\|^\beta) \leq \rho\|ut\|^\beta. \qquad (6)$$

Consequently, we have

$$p_{\overrightarrow{YY}_k(V)}(u,t) \leq \delta \|uv\|^\beta + \rho \|vt\|^\beta < \rho \|ut\|^\beta. \tag{7}$$

In other words, the claims hold for this subcase as long as $\delta < \rho$.

*Subcase 1.2*: the angle $\angle uvt$ is acute. Then we have

$$\|vt\| \leq 2\sin\frac{\theta}{2}\|ut\| = 2\sin\frac{\pi}{k}\|ut\|.$$

Consequently, $\delta\|uv\|^\beta + \rho\|vt\|^\beta$ is at most

$$\delta\|ut\|^\beta + \rho\|vt\|^\beta \leq \delta\|ut\|^\beta + \rho\left(2\sin\frac{\pi}{k}\|ut\|\right)^\beta = \left(\delta + \rho\left(2\sin\frac{\pi}{k}\right)^\beta\right)\|ut\|^\beta. \tag{8}$$

Therefore, if

$$\delta + \rho\left(2\sin\frac{\pi}{k}\right)^\beta \leq \rho$$

then we have

$$p_{\overrightarrow{YY}_k(V)}(u,t) \leq \delta\|uv\|^\beta + \rho\|vt\|^\beta \leq \left(\delta + \rho\left(2\sin\frac{\pi}{k}\right)^\beta\right)\|ut\|^\beta \leq \rho\|ut\|^\beta. \tag{9}$$

In other words, the claims hold for this subcase.

*Case 2*: a link $\overrightarrow{uv}$ with rank $r+1$ does belong to $\overrightarrow{YG}_k(V)$. Then we know that there is a directed path $\Pi_{\overrightarrow{YY}_k}(u,v) = v_1 v_2 \cdots v_h$ from $u$ to $v$ in $\overrightarrow{YY}_k(V)$, where $v_1 = u$ and $v_h = v$. Let $w = v_{h-1}$. If $w$ is $u$ then we have

$$p_{\overrightarrow{YY}_k(V)}(u,v) \leq \|uv\|^\beta < \delta\|uv\|^\beta. \tag{10}$$

So the claims hold. Otherwise, because the directed links $\overrightarrow{uv}$ and $\overrightarrow{wv}$ are at a same cone centered at $v$,

$$rank(\overrightarrow{wv}) < rank(\overrightarrow{uv})$$

due to the selection method in the second phase. Notice that $\angle uvw < \frac{2\pi}{k}$, we have $\|uw\| < \|uv\|$. So,

$$rank(uw) < rank(\overrightarrow{uv}) = r+1.$$

Then by induction, the minimum power consumption path in $\overrightarrow{YY}_k(V)$ connecting $u$ and $w$ is bounded by $\rho\|uw\|^\beta$. Therefore there is a path in $\overrightarrow{YY}_k(V)$ from $u$ to $v$ with the minimum power consumption at most $\rho\|uw\|^\beta + \|wv\|^\beta$. Notice we want to show

$$p_{\overrightarrow{YY}_k(V)}(u,v) \leq \delta\|uv\|^\beta.$$

Again, we have two cases here: whether the angle $\angle uwv$ is acute or not.

*Subcase 2.1*: the angle $\angle uwv$ is not acute. Then we have

$$\|uw\|^\beta + \|wv\|^\beta \leq \|uv\|^\beta.$$

It implies that

$$\rho\|uw\|^\beta + \|wv\|^\beta \leq \rho(\|uv\|^\beta - \|wv\|^\beta) + \|wv\|^\beta = \rho\|uv\|^\beta - (\rho-1)\|wv\|^\beta. \tag{11}$$

¿From the property of civilized graph, we know $\|wv\|^\beta \geq \lambda^\beta$. Then

$$p_{\overrightarrow{YY}_k(V)}(u,v) \leq \rho\|uv\|^\beta - (\rho-1)\lambda^\beta.$$

Therefore, if

$$\rho - \delta \leq (\rho-1)\lambda^\beta$$

we have $p_{\overrightarrow{YY}_k(V)}(u,v)$ is at most

$$\rho\|uv\|^\beta - (\rho-1)\lambda^\beta \leq \rho\|uv\|^\beta - (\rho-\delta) < \rho\|uv\|^\beta - (\rho-\delta)\|uv\|^\beta \leq \delta\|uv\|^\beta. \quad (12)$$

In other words, the claims hold for this subcase.

*Subcases 2.2*: the angle $\angle uwv$ is acute. Then we have

$$\|uw\| \leq 2\sin\frac{\theta}{2}\|uv\| = 2\sin\frac{\pi}{k}\|uv\|.$$

Consequently, $\rho\|uw\|^\beta + \|wv\|^\beta$ is at most

$$\rho\|uw\|^\beta + \|uv\|^\beta \leq \rho\left(2\sin\frac{\pi}{k}\|uv\|\right)^\beta + \|uv\|^\beta = \left(\rho\left(2\sin\frac{\pi}{k}\right)^\beta + 1\right)\|uv\|^\beta. \quad (13)$$

Therefore, if

$$\rho\left(2\sin\frac{\pi}{k}\right)^\beta + 1 \leq \delta$$

then we have

$$p_{\overrightarrow{YY}_k(V)}(u,t) \leq \rho\|uw\|^\beta + \|wv\|^\beta \leq \left(\rho\left(2\sin\frac{\pi}{k}\right)^\beta + 1\right)\|uv\|^\beta \leq \delta\|uv\|^\beta. \quad (14)$$

In other words, the claims hold for this subcase.

As a summary, if

$$\begin{cases} \delta + \rho\left(2\sin\frac{\pi}{k}\right)^\beta \leq \rho, \\ \rho\left(2\sin\frac{\pi}{k}\right)^\beta + 1 \leq \delta, \\ \rho - (\rho-1)\lambda^\beta \leq \delta, \end{cases}$$

then the claims hold for all the cases.

So now we consider whether there exist the constants $\rho$, $\delta$, $k$ and $\lambda$ which make these conditions hold. First assume $\alpha = \left(2\sin\frac{\pi}{k}\right)^\beta$, we need

$$\begin{cases} \delta + \rho\alpha \leq \rho, \\ \rho\alpha + 1 \leq \delta, \\ \rho - (\rho-1)\lambda^\beta \leq \delta, \end{cases}$$

to hold. If $1-\alpha > 0$ and $1-\lambda^\beta > 0$, we can transfer these conditions to

$$\frac{\delta}{1-\alpha} \leq \rho \leq \min\left(\frac{\delta-1}{\alpha}, \frac{\delta-\lambda^\beta}{1-\lambda^\beta}\right).$$

So for a given small $\lambda$, if we select $k$ such that $\alpha = \left(2\sin\frac{\pi}{k}\right)^\beta < \lambda^\beta$ then the existence of $\delta$ and $\rho$ is guaranteed. For example, we can choose $\alpha = \lambda^\beta/2$, then

$\delta = \frac{2-\lambda^\beta}{2-2\lambda^\beta}$ and $\rho = \frac{\delta}{1-\alpha} = 2$. Then we can get the bounded stretch factor. This finishes the proof of the theorem. $\square$

Here we only prove the spanner property of $\overrightarrow{YY}_k(V)$ in civilized graph. Our experimental results show that this sparse topology has a small power stretch factor in practice (see Section 5). We conjecture that $\overrightarrow{YY}_k(V)$ also has a constant bounded power stretch factor theoretically in any unit disk graph. The proof of this conjecture or the construction of a counter-example remain a future work.

## 5. Experiments

In this section we measure the performances of the new sparse and power efficient topologies by conducting some experiments. In a wireless network, each node is expected to potentially send and receive messages from many nodes. In Section 3, we already know that $YG_k^*(V)$, $YY_k(V)$ and $YS_k(V)$ are strongly connected if $UDG(V)$ is connected. So in our experiments, we randomly generate a set $V$ of $n$ wireless nodes and its $UDG(V)$, and test the connectivity of $UDG(V)$. If it is strongly connected, we construct different topologies from $V$. Then we measure the sparseness and the power efficiency of these topologies by the following criteria: the average and the maximum node degree, and the average and the maximum length and power stretch factor. In the experimental results presented here, we generate 100 randomly and uniformly distributed wireless nodes in a $10 \times 10$ square; the number of cones is set to 8 when we construct $YG_k(V)$, $YG_k{}^*(V)$, $YY_k(V)$ and $YS_k(V)$; the power attenuation constant $\beta = 2$; the transmission range is set as $\sqrt{10}$. We generate 500 node sets $V$ (each with 100 nodes) and then generate the graphs for each of these 500 node sets. The average and the maximum are computed over all these 500 node sets. Figure 7 gives all six different topologies for the $UDG(V)$ illustrated by the first figure of Figure 7.

### *5.1. Node Degree*

The node degree of the wireless networks should not be too large. Otherwise a node with a large degree has to communicate with many nodes directly. This increases the interference and the overhead at this node. The node degree should also not be too small either: a small node degree usually implies that the network has a lower fault tolerance and it also tends to increase the overall network power consumption as longer paths may have to be taken. Kleinrock and Silvester [11] showed that the average node degree should be around 6 to gain maximum throughput. Thus, the node degree is an important performance metric for the wireless network topology.

The node degrees of each topology are shown in Table 1. Here $d_{avg}/d_{max}$ is the average/maximum node degree; $I_{avg}/I_{max}$ is the average/maximum node in-degree; $O_{avg}/O_{max}$ is the average/maximum node out-degree. Notice that for a directed graph, its $I_{avg}$ equals to its $O_{avg}$. It shows that $YY_k(V)$ and $YS_k(V)$ have much less number of edges than $YG_k(V)$ and $YG_k{}^*(V)$. In other words, these graphs are sparser, which is also verified by Figure 7. Notice that theoretically, $YY_k(V)$,

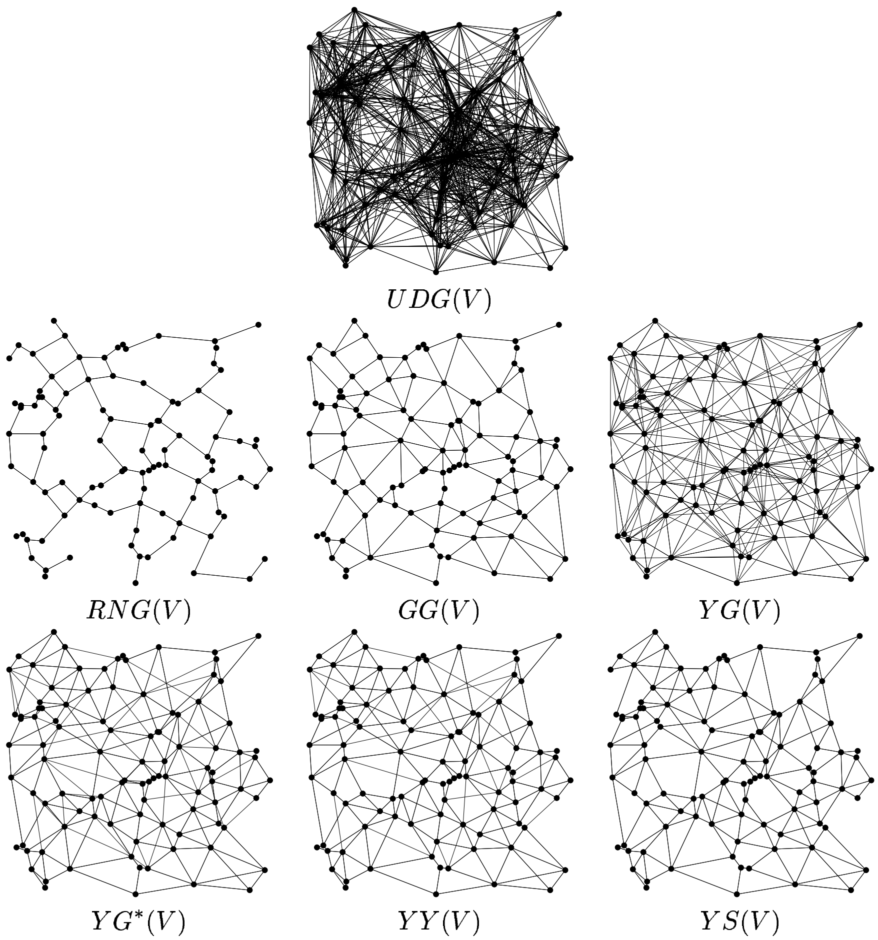


Figure 7: Different topologies from $UDG(V)$.

${YG_k}^*(V)$ and $YS_k(V)$ have bounded node in-degree and out-degree, which can be shown by the maximum node degrees in Table 1.

### *5.2. Stretch Factor*

Besides strong connectivity, the most important design metric of wireless networks is perhaps the power efficiency, as it directly affects both the node and the network lifetime. Table 5.2 summarizes our experimental results of the length and power stretch factors of all these topologies. Here, $t_{avg}/t_{max}$ is the average/maximum length stretch factor; $\rho_{avg}/\rho_{max}$ is the average/maximum power stretch factor. Remember that we only proved that if $\alpha = \left(2\sin\frac{\pi}{k}\right)^{\beta} < \lambda^{\beta}$, *i.e.*, $2\sin\frac{\pi}{k} < \lambda$, $YY_k(V)$ has a bounded stretch factor. Thus, usually $k$ is very large if $\lambda$ is small. However, in our experiments we choose $k = 8$ and $YY_k(V)$ still has small stretch factors. It is not surprise that the average and the maximum node degree of the new topology $YY_k(V)$ are smaller than those of $YG_k(V)$ and ${YG_k}^*(V)$.

| | $d_{avg}$ | $d_{max}$ | $I_{avg}$ | $I_{max}$ | $O_{avg}$ | $O_{max}$ |
|---|---|---|---|---|---|---|
| $UDG$ | 23.57 | 48 | - | - | - | - |
| $RNG$ | 2.38 | 5 | - | - | - | - |
| $GG$ | 3.57 | 8 | - | - | - | - |
| $YG$ | 9.05 | 20 | 6.67 | 18 | 6.67 | 8 |
| $YG^*$ | 5.29 | 10 | 4.79 | 10 | 4.79 | 8 |
| $YY$ | 5.02 | 9 | 4.65 | 8 | 4.65 | 8 |
| $YS$ | 4.28 | 8 | - | - | - | - |

Table 1: Node degrees of different topologies.

However, it is surprise that the average and the maximum power stretch factors of $YY_k(V)$ are at the same level of those of the $YG_k(V)$ and $YG_k{}^*(V)$. In addition, though $YS_k(V)$ is not a spanner for length and power, it has small stretch factors in practice.

| | $t_{avg}$ | $t_{max}$ | $\rho_{avg}$ | $\rho_{max}$ |
|---|---|---|---|---|
| $UDG$ | 1.000 | 1.000 | 1.000 | 1.000 |
| $RNG$ | 1.319 | 4.549 | 1.056 | 3.509 |
| $GG$ | 1.124 | 1.991 | 1.000 | 1.000 |
| $YG$ | 1.041 | 1.723 | 1.002 | 1.461 |
| $YG^*$ | 1.070 | 1.895 | 1.003 | 1.461 |
| $YY$ | 1.074 | 1.895 | 1.004 | 1.461 |
| $YS$ | 1.090 | 2.174 | 1.004 | 1.473 |

Table 2: Length (Power) stretch factors.

## 6. Conclusion

In this paper, we presented several efficient localized algorithms to construct network topologies with bounded node degrees for wireless ad hoc networks. We showed that $\overrightarrow{YG}_k^*(V)$, $\overrightarrow{YY}_k(V)$ have bounded power stretch factors, while $YS_k(V)$ does not have. We summarize the properties of these topologies in Table 6, and the relations among all these subgraphs of $UDG(V)$ as follows.

$$RNG(V) \subseteq YS_k(V) \subseteq YY_k(V) \subseteq YG_k^*(V) \subseteq UDG(V), YY_k(V) \subseteq YG_k(V). \quad (15)$$

Notice $YG_k^*(V)$ may not be a subgraph of $YG_k(V)$.

Until now, we always assumed that the wireless devices are static or quasi-static in a reasonable time period. It is not difficult to update these graphs when the wireless nodes are moving because whether an edge $uv$ is in a subgraph discussed here can be decided locally using only the one-hop neighbors of a node.

Notice that, even the graph $\overrightarrow{YY}_k(V)$ has a good power stretch factor in practice for randomly and uniformly distributed nodes and a bounded power stretch factor in civilized graph. It is still an open problem whether it is a spanner theoretically

| | Power Stretch Factor | Length Stretch Factor | Max Node Degree |
|---|---|---|---|
| $UDG$ | 1 | 1 | $n-1$ |
| $RNG$ | $n-1$ | $O(n)$ | $n-1$ |
| $GG$ | 1 | $O(\sqrt{n})$ | $n-1$ |
| $YG$ | $\frac{1}{1-(2\sin\frac{\pi}{k})^\beta}$ | $\frac{1}{1-2\sin\frac{\pi}{k}}$ | $n-1$ |
| $YG^*$ | $\left(\frac{1}{1-(2\sin\frac{\pi}{k})^\beta}\right)^2$ | $\left(\frac{1}{1-2\sin\frac{\pi}{k}}\right)^2$ | $(k+1)^2-1$ |
| $YY$ | $2^\diamond$ | ? | $2k$ |
| $YS$ | $O(n)$ | $O(n)$ | $k$ |

Table 3: The power stretch factor and the maximum node degree of these graphs. Here, the number with $^\diamond$ is only true in civilized graph.

in general unit disk graph. We conjecture that it is a spanner and leave the proof or the construction of a counter-example as a future work.


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