Natural Semantics

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In this lecture we demonstrate another kind of proof tree. Instead of describing types, it describes the value of an expression.

- the different parts of a natural rule
- how to structure a proof-tree
- how natural semantics are different than type semantics

How do you explain what an expression means?

- The major insight is that *the meaning of an expression can be determined by combining the meaning of its subexpressions.*

Examples:

- What is the meaning of `if true then 4 else 38`?
- What is the meaning of `8 + 34`?

Evaluations

An *evaluation* has the following form:

$$e \Downarrow v$$

where *e* is some expression, and v is a *value*.

- `if true then` 4 `else` $38 \Downarrow$ *4*
- `13` $+$ `4` $\Downarrow$ *17*

Note: a textual number `13` represents the mathematical number *13*.

The Parts of a Rule

Assumptions

$$\frac{M_1 \Downarrow \textit{true} \qquad M_2 \Downarrow V}{\texttt{if } M_1 \texttt{ then } M_2 \texttt{ else } M_3 \Downarrow V}$$

Conclusion

- If a rule has no assumptions, then it is called an *axiom*.
- The $\Downarrow$ is pronounced “evaluates to” or “down-arrow”.

Some Simple Rules

Integers

$$\frac{}{n \Downarrow n}$$

Parenthesis

$$\frac{e \Downarrow V}{(e) \Downarrow V}$$

Booleans

$$\frac{}{\texttt{true} \Downarrow true} \qquad \frac{}{\texttt{false} \Downarrow false}$$

Addition

$$\frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2}{e_1{+}e_2 \Downarrow v_1 + v_2}$$

General Operators

$$\frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2}{e_1 \oplus e_2 \Downarrow v_1 \oplus v_2}$$

To use these rules, you combine them into a *proof tree*.
To prove: `(2+3) * (3+5)` $\Downarrow$ *40*

$$\dfrac{\dfrac{\dfrac{\dfrac{}{3 \Downarrow 3} \quad \dfrac{}{2 \Downarrow 2}}{2+3 \Downarrow 5}}{(2+3) \Downarrow 5} \qquad \dfrac{\dfrac{\dfrac{}{3 \Downarrow 3} \quad \dfrac{}{5 \Downarrow 5}}{3+5 \Downarrow 8}}{(3+5) \Downarrow 8}}{(2+3) * (3+5) \Downarrow 40}$$

- By the way, this lecture is about recursion. What are the base cases?

if

$$\frac{e_1 \Downarrow \textit{true} \quad e_2 \Downarrow V}{\texttt{if } e_1 \texttt{ then } e_2 \texttt{ else } e_3 \Downarrow V} \qquad \frac{e_1 \Downarrow \textit{false} \quad e_3 \Downarrow V}{\texttt{if } e_1 \texttt{ then } e_2 \texttt{ else } e_3 \Downarrow V}$$

and

$$\frac{e_1 \Downarrow \textit{true} \quad e_2 \Downarrow v_2}{e_1 \;\&\&\; e_2 \Downarrow v_2} \qquad \frac{e_1 \Downarrow \textit{false}}{e_1 \;\&\&\; e_2 \Downarrow \textit{false}}$$

or

$$\frac{e_1 \Downarrow \textit{false} \quad e_2 \Downarrow v_2}{e_1 \;||\; e_2 \Downarrow v_2} \qquad \frac{e_1 \Downarrow \textit{true}}{e_1 \;||\; e_2 \Downarrow \textit{true}}$$

How to do it

1. Draw a line over the term
2. Pick the applicable rule from your list
3. Use the rule to place the assumptions, if any
4. Recurse over the assumptions.

Example: show that

$$\texttt{if false || true then } 40 \texttt{ else } 50 \Downarrow \textit{40}$$

Step 1: Draw a line over the term.

$$\overline{\texttt{if false || true then } 40 \texttt{ else } 50 \Downarrow \mathit{40}}$$

- Step 2: Pick the applicable rule.
- The applicable rule will be the one that matches the type of the "outermost part" of the expression. This one is an `if` expression, so we'll use the rules that apply to `if`.
- There are two `if` rules; we know we need the one that has the conditional evaluating to *true*.

Step 3: Use the rule to place the assumptions

$$\frac{\texttt{false || true} \Downarrow \textit{true} \quad 40 \Downarrow \textit{40}}{\texttt{if false || true then } 40 \texttt{ else } 50 \Downarrow \textit{40}}$$

Step 4: Recurse....

$$\frac{\dfrac{\texttt{false} \Downarrow \textit{false} \quad \texttt{true} \Downarrow \textit{true}}{\texttt{false || true} \Downarrow \textit{true}} \quad 40 \Downarrow \textit{40}}{\texttt{if false || true then } 40 \texttt{ else } 50 \Downarrow \textit{40}}$$

Rinse, lather, repeat...

$$\dfrac{\dfrac{\dfrac{}{\texttt{false} \Downarrow \mathit{false}} \qquad \dfrac{}{\texttt{true} \Downarrow \mathit{true}}}{\texttt{false || true} \Downarrow \mathit{true}} \qquad \dfrac{}{40 \Downarrow \mathit{40}}}{\texttt{if false || true then } 40 \texttt{ else } 50 \Downarrow \mathit{40}}$$

- You are done when every expresssion has a line over it.

Variables

- Natural semantics does not use an environment.
- To express the meaning of variable substitution, we use a fraction-like notation.
- $[e_1/\mathtt{x}]e_2$ means “Replace all occurances of x in e_2 with e_1.”
- So, $[3/\mathtt{x}](2+\mathtt{x}) \Rightarrow (2+3)$

Compare with

$$\frac{a}{x} \times (bx + cx) = (ba + ca)$$

Rules involving variables

functions

$$\frac{}{\texttt{fun x} \mathrel{-\!\!>} e_1 \Downarrow \texttt{fun x} \mathrel{-\!\!>} e_1}$$

application

$$\frac{e_1 \Downarrow \texttt{fun x} \mathrel{-\!\!>} e_2 \quad [e_3/\mathsf{x}]e_2 \Downarrow V}{e_1\ e_3 \Downarrow V}$$

let

$$\frac{e_3 \Downarrow U \quad [U/\mathsf{x}]e_2 \Downarrow V}{\texttt{let x} = e_3\ \texttt{in}\ e_2 \Downarrow V}$$

Example

Show that `let f = fun x -> x + 2 in f (f 3)` $\Downarrow 7$.

- Step one: pick the appropriate rule.

$$\frac{(\texttt{fun}\ \texttt{x}\texttt{->}\texttt{x}+2)((\texttt{fun}\ \texttt{x}\texttt{->}\texttt{x}+2)\ 3) \Downarrow 7}{\texttt{let}\ \texttt{f} = \texttt{fun}\ \texttt{x}\ \texttt{->}\texttt{x}+2\ \texttt{in}\ \texttt{f}(\texttt{f}\ 3) \Downarrow 7}$$

- Here we’ve used the **let** rule. We’ve not written down the fraction here, instead we’ve expanded it out.
- What should the next rule be?

Example, part 2

- There are two functions you could use here. Which to chose?
- Answer: Always pick the *outermost expression*.

$$\cfrac{\cfrac{\texttt{fun}\ \mathtt{x}{-}{>}\mathtt{x}+2 \Downarrow \texttt{fun}\ \mathtt{x}{-}{>}\mathtt{x}+2 \qquad ((\texttt{fun}\ \mathtt{x}{-}{>}\mathtt{x}+2)\ 3)+2 \Downarrow 7}{(\texttt{fun}\ \mathtt{x}{-}{>}\mathtt{x}+2)((\texttt{fun}\ \mathtt{x}{-}{>}\mathtt{x}+2)\ 3) \Downarrow 7}}{\texttt{let}\ \mathtt{f} = \texttt{fun}\ \mathtt{x}\ {-}{>}\mathtt{x}+2\ \texttt{in}\ \mathtt{f}(\mathtt{f}\ 3) \Downarrow 7}$$

- Next: there are two rules.

Example, part 3

$$\dfrac{\dfrac{}{\texttt{fun x->x}+2 \Downarrow \texttt{fun x->x}+2} \qquad \dfrac{(\texttt{fun x->x}+2)\ 3 \Downarrow 5 \quad 2 \Downarrow 2}{((\texttt{fun x->x}+2)\ 3)+2 \Downarrow 7}}{\dfrac{(\texttt{fun x->x}+2)((\texttt{fun x->x}+2)\ 3) \Downarrow 7}{\texttt{let f = fun x ->x}+2\ \texttt{in f(f 3)} \Downarrow 7}}$$

- Use the function rule as an axiom.
- Use the function application rule again for the inner application.

Example, part 4

$$\dfrac{\dfrac{}{\texttt{fun x->x}+2 \Downarrow \texttt{fun x->x}+2} \qquad \dfrac{\dfrac{\texttt{fun x->x}+2 \Downarrow \texttt{fun x->x}+2 \qquad 3+2 \Downarrow 5}{(\texttt{fun x->x}+2)\ 3 \Downarrow 5}}{((\texttt{fun x->x}+2)\ 3)+2 \Downarrow}}{\dfrac{(\texttt{fun x->x}+2)((\texttt{fun x->x}+2)\ 3) \Downarrow 7}{\texttt{let f} = \texttt{fun x ->x}+2\ \texttt{in f(f 3)} \Downarrow 7}}$$

- Now there are two axioms left, and a plus rule.

Example, part 5

$$
\dfrac{\dfrac{}{\texttt{fun x->...} \Downarrow \ldots} \qquad \dfrac{\dfrac{\dfrac{}{\texttt{fun x->...} \Downarrow \ldots} \qquad \dfrac{\dfrac{}{3 \Downarrow \mathit{3}} \quad \dfrac{}{2 \Downarrow \mathit{2}}}{3 + 2 \Downarrow \mathit{5}}}{(\texttt{fun x->x} + 2)\ 3 \Downarrow \mathit{5}} \qquad \dfrac{}{2 \Downarrow \mathit{2}}}{((\texttt{fun x->x} + 2)\ 3) + 2 \Downarrow \mathit{7}}}{\dfrac{(\texttt{fun x->x} + 2)((\texttt{fun x->x} + 2)\ 3) \Downarrow \mathit{7}}{\texttt{let f = fun x ->x} + 2\ \texttt{in f(f 3)} \Downarrow \mathit{7}}}
$$

- Note: you can really think of the function rule as a kind of constant rule.
- The ... is because there's not enough space on the slide.

Try the following problems.

1. (120) Show that $\texttt{let}\ \mathtt{x} = 5\ \texttt{in}\ \texttt{let}\ \mathtt{y} = 10\ \texttt{in}\ x + y \Downarrow$ *15*
2. (121) Show that $\texttt{let}\ \mathtt{x} = 5\ \texttt{in}\ \texttt{let}\ \mathtt{y} = 10\ \texttt{in}\ x\ * x \Downarrow$ *25*
3. (122) Show that $\texttt{let}\ \mathtt{f} = \texttt{fun}\ \mathtt{x}{-}{>}\mathtt{x} * \mathtt{x}\ \texttt{in}\ \texttt{let}\ \mathtt{y} = 4\ \texttt{in}\ \mathtt{f}\ \mathtt{y} \Downarrow$ *16*

Problem 1

Show that $\texttt{let}\ \mathtt{x} = 5\ \texttt{in}\ \texttt{let}\ \mathtt{y} = 10\ \texttt{in}\ x + y \Downarrow \mathit{15}$

Solution:

$$\dfrac{\dfrac{}{5 \Downarrow \mathit{5}} \qquad \dfrac{\dfrac{}{\mathtt{y} \Downarrow \mathit{10}} \qquad \dfrac{\dfrac{}{5 \Downarrow \mathit{5}} \qquad \dfrac{}{10 \Downarrow \mathit{10}}}{5 + 10 \Downarrow \mathit{15}}}{\texttt{let}\ 10 = 10\ \texttt{in}\ 5 + y \Downarrow \mathit{15}}}{\texttt{let}\ \mathtt{x} = 5\ \texttt{in}\ \texttt{let}\ \mathtt{y} = 10\ \texttt{in}\ x + y \Downarrow \mathit{15}}$$

Problem 2

Show that $\texttt{let}\ \mathtt{x} = 5\ \texttt{in}\ \texttt{let}\ \mathtt{y} = 10\ \texttt{in}\ x\ * x \Downarrow \mathit{25}$

Solution:

$$\dfrac{\dfrac{}{5 \Downarrow \mathit{5}} \qquad \dfrac{\dfrac{}{10 \Downarrow \mathit{10}} \qquad \dfrac{\dfrac{}{5 \Downarrow \mathit{5}} \qquad \dfrac{}{5 \Downarrow \mathit{5}}}{5 * 5 \Downarrow \mathit{25}}}{\texttt{let}\ \mathtt{y} = 10\ \texttt{in}\ 5\ * 5 \Downarrow \mathit{25}}}{\texttt{let}\ \mathtt{x} = 5\ \texttt{in}\ \texttt{let}\ \mathtt{y} = 10\ \texttt{in}\ x\ * x \Downarrow \mathit{25}}$$

Show that $\texttt{let}\ \mathtt{f} = \texttt{fun}\ \mathtt{x}{-}{>}\mathtt{x} * \mathtt{x}\ \texttt{in}\ \texttt{let}\ \mathtt{y} = 4\ \texttt{in}\ \mathtt{f}\ \mathtt{y} \Downarrow \mathit{16}$

Solution:

$$
\dfrac{\dfrac{}{\ldots \Downarrow \ldots} \qquad \dfrac{\dfrac{}{4 \Downarrow \mathit{4}} \qquad \dfrac{\dfrac{}{\texttt{fun}\ \mathtt{x}{-}{>}\mathtt{x} * \mathtt{x} \Downarrow \texttt{fun}\ \mathtt{x}{-}{>}\mathtt{x} * \mathtt{x}} \qquad \dfrac{\dfrac{}{4 \Downarrow \mathit{4}} \quad \dfrac{}{4 \Downarrow \mathit{4}}}{4 * 4 \Downarrow \mathit{16}}}{(\texttt{fun}\ \mathtt{x}{-}{>}\mathtt{x} * \mathtt{x})\ 4 \Downarrow \mathit{16}}}{\texttt{let}\ \mathtt{y} = 4\ \texttt{in}\ (\texttt{fun}\ \mathtt{x}{-}{>}\mathtt{x} * \mathtt{x})\ \mathtt{y} \Downarrow \mathit{16}}}{\texttt{let}\ \mathtt{f} = \texttt{fun}\ \mathtt{x}{-}{>}\mathtt{x} * \mathtt{x}\ \texttt{in}\ \texttt{let}\ \mathtt{y} = 4\ \texttt{in}\ \mathtt{f}\ \mathtt{y} \Downarrow \mathit{16}}
$$