Natural Semantics

Mattox Beckman

beckman@iit.edu

Illinois Institute of Technology

Objectives

In this lecture we demonstrate another kind of proof tree. Instead of describing types, it describes the value of an expression.

- the different parts of a natural rule
- how to structure a proof-tree
- how natural semantics are different than type semantics

Meaning

How do you explain what an expression means?

- The major insight is that *the meaning of an expression can be determined by combining the meaning of its subexpressions.*

Examples:

- What is the meaning of `if true then 4 else 38`?
- What is the meaning of `8 + 34`?

Evaluations

An *evaluation* has the following form:

$$e \Downarrow v$$

where *e* is some expression, and v is a *value*.

- `if` `true` `then` 4 `else` $38 \Downarrow$ *4*
- $13 + 4 \Downarrow$ *17*

Note: a textual number 13 represents the mathematical number *13*.

The Parts of a Rule

Assumptions

$$\frac{M_1 \Downarrow \textit{true} \qquad M_2 \Downarrow V}{\texttt{if}\ M_1\ \texttt{then}\ M_2\ \texttt{else}\ M_3 \Downarrow V}$$

Conclusion

- If a rule has no assumptions, then it is called an *axiom*.
- The $\Downarrow$ is pronounced “evaluates to” or “down-arrow”.

Some Simple Rules

Integers

$$\frac{}{n \Downarrow n}$$

Parenthesis

$$\frac{e \Downarrow V}{(e) \Downarrow V}$$

Booleans

$$\frac{}{\texttt{true} \Downarrow \mathit{true}} \qquad \frac{}{\texttt{false} \Downarrow \mathit{false}}$$

Addition

$$\frac{e_1 \Downarrow v_1 \qquad e_2 \Downarrow v_2}{e_1{+}e_2 \Downarrow v_1 + v_2}$$

General Operators

$$\frac{e_1 \Downarrow v_1 \qquad e_2 \Downarrow v_2}{e_1 \oplus e_2 \Downarrow v_1 \oplus v_2}$$

Proof Trees

To use these rules, you combine them into a *proof tree*.
To prove: `(2+3) * (3+5)` $\Downarrow$ *40*

$$\dfrac{\dfrac{\dfrac{\dfrac{}{3 \Downarrow \textit{3}} \qquad \dfrac{}{2 \Downarrow \textit{2}}}{2+3 \Downarrow \textit{5}}}{(2+3) \Downarrow \textit{5}} \qquad \dfrac{\dfrac{\dfrac{}{3 \Downarrow \textit{3}} \qquad \dfrac{}{5 \Downarrow \textit{5}}}{3+5 \Downarrow \textit{8}}}{(3+5) \Downarrow \textit{8}}}{(2+3) * (3+5) \Downarrow \textit{40}}$$

- By the way, this lecture is about recursion. What are the base cases?

if, and, or

if

$$\frac{e_1 \Downarrow \textit{true} \qquad e_2 \Downarrow V}{\texttt{if}\ e_1\ \texttt{then}\ e_2\ \texttt{else}\ e_3 \Downarrow V} \qquad \frac{e_1 \Downarrow \textit{false} \qquad e_3 \Downarrow V}{\texttt{if}\ e_1\ \texttt{then}\ e_2\ \texttt{else}\ e_3 \Downarrow V}$$

and

$$\frac{e_1 \Downarrow \textit{true} \qquad e_2 \Downarrow v_2}{e_1\ \&\&\ e_2 \Downarrow v_2} \qquad \frac{e_1 \Downarrow \textit{false}}{e_1\ \&\&\ e_2 \Downarrow \textit{false}}$$

or

$$\frac{e_1 \Downarrow \textit{false} \qquad e_2 \Downarrow v_2}{e_1\ ||\ e_2 \Downarrow v_2} \qquad \frac{e_1 \Downarrow \textit{true}}{e_1\ ||\ e_2 \Downarrow \textit{true}}$$

How to do it

1. Draw a line over the term
2. Pick the applicable rule from your list
3. Use the rule to place the assumptions, if any
4. Recurse over the assumptions.

Example: show that

$$\texttt{if false || true then } 40 \texttt{ else } 50 \Downarrow \mathit{40}$$

Step 1: Draw a line over the term.

$$\overline{\texttt{if false || true then } 40 \texttt{ else } 50 \Downarrow \mathit{40}}$$

- Step 2: Pick the applicable rule.
- The applicable rule will be the one that matches the type of the “outermost part” of the expression. This one is an `if` expression, so we’ll use the rules that apply to `if`.
- There are two `if` rules; we know we need the one that has the conditional evaluating to *true*.

Step 3: Use the rule to place the assumptions

$$\frac{\texttt{false || true} \Downarrow \mathit{true} \qquad 40 \Downarrow \mathit{40}}{\texttt{if false || true then } 40 \texttt{ else } 50 \Downarrow \mathit{40}}$$

Step 4: Recurse....

$$\frac{\dfrac{\texttt{false} \Downarrow \mathit{false} \qquad \texttt{true} \Downarrow \mathit{true}}{\texttt{false || true} \Downarrow \mathit{true}} \qquad 40 \Downarrow \mathit{40}}{\texttt{if false || true then } 40 \texttt{ else } 50 \Downarrow \mathit{40}}$$

Rinse, lather, repeat...

$$\frac{\dfrac{\overline{\texttt{false} \Downarrow \mathit{false}} \qquad \overline{\texttt{true} \Downarrow \mathit{true}}}{\texttt{false || true} \Downarrow \mathit{true}} \qquad \overline{40 \Downarrow \mathit{40}}}{\texttt{if false || true then } 40 \texttt{ else } 50 \Downarrow \mathit{40}}$$

- You are done when every expresssion has a line over it.

Variables

- Natural semantics does not use an environment.
- To express the meaning of variable substitution, we use a fraction-like notation.
- $[e_1/\mathtt{x}]e_2$ means "Replace all occurances of x in e_2 with e_1."
- So, $[3/\mathtt{x}](2+\mathtt{x}) \Rightarrow (2+3)$

Compare with

$$\frac{a}{x} \times (bx + cx) = (ba + ca)$$

Rules involving variables

functions

$$\frac{}{\mathtt{fun}\ \mathtt{x} \mathrel{-\!\!>} e_1 \Downarrow \mathtt{fun}\ \mathtt{x} \mathrel{-\!\!>} e_1}$$

application

$$\frac{e_1 \Downarrow \mathtt{fun}\ \mathtt{x} \mathrel{-\!\!>} e_2 \qquad [e_3/\mathtt{x}]e_2 \Downarrow V}{e_1\ e_3 \Downarrow V}$$

let

$$\frac{e_3 \Downarrow U \qquad [U/\mathtt{x}]e_2 \Downarrow V}{\mathtt{let}\ \mathtt{x} = e_3\ \mathtt{in}\ e_2 \Downarrow V}$$

Example

Show that `let f = fun x -> x + 2 in f (f 3)` $\Downarrow$ *7*.

- Step one: pick the appropriate rule.

$$\frac{(\mathtt{fun}\ \mathtt{x} \mathrel{-\!\!>} \mathtt{x}+2)((\mathtt{fun}\ \mathtt{x} \mathrel{-\!\!>} \mathtt{x}+2)\ 3) \Downarrow 7}{\mathtt{let}\ \mathtt{f} = \mathtt{fun}\ \mathtt{x} \mathrel{-\!\!>} \mathtt{x}+2\ \mathtt{in}\ \mathtt{f}(\mathtt{f}\ 3) \Downarrow 7}$$

- Here we've used the **let** rule. We've not written down the fraction here, instead we've expanded it out.
- What should the next rule be?

Example, part 2

- There are two functions you could use here. Which to chose?
- Answer: Always pick the *outermost expression*.

$$\cfrac{\mathtt{fun}\ \mathtt{x} \mathrel{-\!\!>} \mathtt{x}+2 \Downarrow \mathtt{fun}\ \mathtt{x} \mathrel{-\!\!>} \mathtt{x}+2 \qquad ((\mathtt{fun}\ \mathtt{x} \mathrel{-\!\!>} \mathtt{x}+2)\ 3)+2 \Downarrow 7}{\cfrac{(\mathtt{fun}\ \mathtt{x} \mathrel{-\!\!>} \mathtt{x}+2)((\mathtt{fun}\ \mathtt{x} \mathrel{-\!\!>} \mathtt{x}+2)\ 3) \Downarrow 7}{\mathtt{let}\ \mathtt{f} = \mathtt{fun}\ \mathtt{x} \mathrel{-\!\!>} \mathtt{x}+2\ \mathtt{in}\ \mathtt{f}(\mathtt{f}\ 3) \Downarrow 7}}$$

- Next: there are two rules.

Example, part 3

$$\dfrac{\dfrac{}{\mathtt{fun}\ \mathtt{x}{-}{>}\mathtt{x}+2 \Downarrow \mathtt{fun}\ \mathtt{x}{-}{>}\mathtt{x}+2} \quad \dfrac{(\mathtt{fun}\ \mathtt{x}{-}{>}\mathtt{x}+2)\ 3 \Downarrow \mathit{5} \quad 2 \Downarrow \mathit{2}}{((\mathtt{fun}\ \mathtt{x}{-}{>}\mathtt{x}+2)\ 3)+2 \Downarrow \mathit{7}}}{\dfrac{(\mathtt{fun}\ \mathtt{x}{-}{>}\mathtt{x}+2)((\mathtt{fun}\ \mathtt{x}{-}{>}\mathtt{x}+2)\ 3) \Downarrow \mathit{7}}{\mathtt{let}\ \mathtt{f} = \mathtt{fun}\ \mathtt{x}\ {-}{>}\mathtt{x}+2\ \mathtt{in}\ \mathtt{f}(\mathtt{f}\ 3) \Downarrow \mathit{7}}}$$

- Use the function rule as an axiom.
- Use the function application rule again for the inner application.

Example, part 4

$$\dfrac{\dfrac{}{\mathtt{fun}\ \mathtt{x}{-}{>}\mathtt{x}+2 \Downarrow \mathtt{fun}\ \mathtt{x}{-}{>}\mathtt{x}+2} \quad \dfrac{\dfrac{\mathtt{fun}\ \mathtt{x}{-}{>}\mathtt{x}+2 \Downarrow \mathtt{fun}\ \mathtt{x}{-}{>}\mathtt{x}+2 \quad 3+2 \Downarrow \mathit{5}}{(\mathtt{fun}\ \mathtt{x}{-}{>}\mathtt{x}+2)\ 3 \Downarrow \mathit{5}}}{((\mathtt{fun}\ \mathtt{x}{-}{>}\mathtt{x}+2)\ 3)+2 \Downarrow \mathit{7}}}{\dfrac{(\mathtt{fun}\ \mathtt{x}{-}{>}\mathtt{x}+2)((\mathtt{fun}\ \mathtt{x}{-}{>}\mathtt{x}+2)\ 3) \Downarrow \mathit{7}}{\mathtt{let}\ \mathtt{f} = \mathtt{fun}\ \mathtt{x}\ {-}{>}\mathtt{x}+2\ \mathtt{in}\ \mathtt{f}(\mathtt{f}\ 3) \Downarrow \mathit{7}}}$$

- Now there are two axioms left, and a plus rule.

Example, part 5

$$\dfrac{\dfrac{}{\mathtt{fun}\ \mathtt{x}{-}{>}\ldots \Downarrow \ldots} \quad \dfrac{\dfrac{\dfrac{}{\mathtt{fun}\ \mathtt{x}{-}{>}\ldots \Downarrow \ldots} \quad \dfrac{\dfrac{}{3 \Downarrow \mathit{3}} \quad \dfrac{}{2 \Downarrow \mathit{2}}}{3+2 \Downarrow \mathit{5}}}{(\mathtt{fun}\ \mathtt{x}{-}{>}\mathtt{x}+2)\ 3 \Downarrow \mathit{5}} \quad \dfrac{}{2 \Downarrow \mathit{2}}}{((\mathtt{fun}\ \mathtt{x}{-}{>}\mathtt{x}+2)\ 3)+2 \Downarrow \mathit{7}}}{\dfrac{(\mathtt{fun}\ \mathtt{x}{-}{>}\mathtt{x}+2)((\mathtt{fun}\ \mathtt{x}{-}{>}\mathtt{x}+2)\ 3) \Downarrow \mathit{7}}{\mathtt{let}\ \mathtt{f} = \mathtt{fun}\ \mathtt{x}\ {-}{>}\mathtt{x}+2\ \mathtt{in}\ \mathtt{f}(\mathtt{f}\ 3) \Downarrow \mathit{7}}}$$

- Note: you can really think of the function rule as a kind of constant rule.
- The ... is because there's not enough space on the slide.

Problems

Try the following problems.

1. (120) Show that $\mathtt{let}\ \mathtt{x} = 5\ \mathtt{in}\ \mathtt{let}\ \mathtt{y} = 10\ \mathtt{in}\ x + y \Downarrow \mathit{15}$
2. (121) Show that $\mathtt{let}\ \mathtt{x} = 5\ \mathtt{in}\ \mathtt{let}\ \mathtt{y} = 10\ \mathtt{in}\ x\ * x \Downarrow \mathit{25}$
3. (122) Show that $\mathtt{let}\ \mathtt{f} = \mathtt{fun}\ \mathtt{x}{-}{>}\mathtt{x}*\mathtt{x}\ \mathtt{in}\ \mathtt{let}\ \mathtt{y} = 4\ \mathtt{in}\ \mathtt{f}\ \mathtt{y} \Downarrow \mathit{16}$

Problem 1

Show that $\texttt{let}\ \mathrm{x} = 5\ \texttt{in}\ \texttt{let}\ \mathrm{y} = 10\ \texttt{in}\ x + y \Downarrow \mathit{15}$

Solution:

$$\frac{\frac{}{5 \Downarrow \mathit{5}} \qquad \frac{\frac{}{\mathrm{y} \Downarrow \mathit{10}} \qquad \frac{\frac{}{5 \Downarrow \mathit{5}} \quad \frac{}{10 \Downarrow \mathit{10}}}{5 + 10 \Downarrow \mathit{15}}}{\texttt{let}\ 10 = 10\ \texttt{in}\ 5 + y \Downarrow \mathit{15}}}{\texttt{let}\ \mathrm{x} = 5\ \texttt{in}\ \texttt{let}\ \mathrm{y} = 10\ \texttt{in}\ x + y \Downarrow \mathit{15}}$$

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Problem 2

§4 **Activity**

Show that $\texttt{let}\ \mathrm{x} = 5\ \texttt{in}\ \texttt{let}\ \mathrm{y} = 10\ \texttt{in}\ x\ * x \Downarrow \mathit{25}$

Solution:

$$\frac{\frac{}{5 \Downarrow \mathit{5}} \qquad \frac{\frac{}{10 \Downarrow \mathit{10}} \qquad \frac{\frac{}{5 \Downarrow \mathit{5}} \quad \frac{}{5 \Downarrow \mathit{5}}}{5 * 5 \Downarrow \mathit{25}}}{\texttt{let}\ \mathrm{y} = 10\ \texttt{in}\ 5\ * 5 \Downarrow \mathit{25}}}{\texttt{let}\ \mathrm{x} = 5\ \texttt{in}\ \texttt{let}\ \mathrm{y} = 10\ \texttt{in}\ x\ * x \Downarrow \mathit{25}}$$

Natural Semantics

Problem 3

§4 **Activity**

Show that $\texttt{let}\ \mathrm{f} = \texttt{fun}\ \mathrm{x}{-}{>}\mathrm{x} * \mathrm{x}\ \texttt{in}\ \texttt{let}\ \mathrm{y} = 4\ \texttt{in}\ \mathrm{f}\ \mathrm{y} \Downarrow \mathit{16}$

Solution:

$$\frac{\frac{}{\ldots \Downarrow \ldots} \qquad \frac{\frac{}{4 \Downarrow \mathit{4}} \qquad \frac{\frac{}{\texttt{fun}\ \mathrm{x}{-}{>}\mathrm{x} * \mathrm{x} \Downarrow \texttt{fun}\ \mathrm{x}{-}{>}\mathrm{x} * \mathrm{x}} \qquad \frac{\frac{}{4 \Downarrow \mathit{4}} \quad \frac{}{4 \Downarrow \mathit{4}}}{4 * 4 \Downarrow \mathit{16}}}{(\texttt{fun}\ \mathrm{x}{-}{>}\mathrm{x} * \mathrm{x})\ 4 \Downarrow \mathit{16}}}{\texttt{let}\ \mathrm{y} = 4\ \texttt{in}\ (\texttt{fun}\ \mathrm{x}{-}{>}\mathrm{x} * \mathrm{x})\ \mathrm{y} \Downarrow \mathit{16}}}{\texttt{let}\ \mathrm{f} = \texttt{fun}\ \mathrm{x}{-}{>}\mathrm{x} * \mathrm{x}\ \texttt{in}\ \texttt{let}\ \mathrm{y} = 4\ \texttt{in}\ \mathrm{f}\ \mathrm{y} \Downarrow \mathit{16}}$$