Illinois Institute of Technology
Department of Computer Science

CS 330 Discrete Structures

Fall Semester, 2010

Homework Assignment 10

Solution

1. (#48 in page 529) Suppose that R and S are reflexive relations on a set A. Prove or disprove each of these statements.

   (a) $R \cup S$ is reflexive

   (b) $R \cap S$ is reflexive

   (c) $R \oplus S$ is irreflexive

   (d) $R - S$ is irreflexive

   (e) $S \circ R$ is reflexive

   Solution:

   (a) Since R contains all the pairs (x, x), so does $R \cup S$. Therefore, $R \cup S$ is reflexive.

   (b) Since R and S each contain all the pairs (x, x), so does $R \cap S$. Therefore, $R \cap S$ is reflexive.

   (c) Since R and S each contain all the pairs (x, x), we know that $R \oplus S$ contains none of these pairs. Therefore $R \oplus S$ is irreflexive.

   (d) Since R and S each contain all the pairs (x, x), we know that $R - S$ contains none of these pairs. Therefore $R - S$ is irreflexive.

   (e) Since R and S each contain all the pairs (x, x), so does $R \circ S$. Therefore, $R \circ S$ is reflexive.

2. (#34 in page 544) Let R be a relation on a set A. Explain how to use the directed graph representing R to obtain the directed graph representing the complementary relation $\bar{R}$.

   Solution: For each pair (a, b) of vertices (including the pairs (a, a) in which the two vertices are the same), if there is an edge from a to b, then erase it, and if there is no edge from a to b, then add it in.

3. (#54 in page 565) Suppose that R_1 and R_2 are equivalence relations on a set A. Let P_1 and P_2 be the partitions that correspond to R_1 and R_2, respectively. Show that $R_1 \subseteq R_2$ if and only if P_1 is a refinement of P_2.

   Solution: First, suppose that $R_1 \subseteq R_2$. We must show that P_1 is a refinement of P_2. Let $[a]_{R_1}$ be an equivalence class in P_1. We must show that $[a]_{R_1}$ is contained in an equivalence class in P_2. In fact, we will show that $[a]_{R_1} \subseteq [a]_{R_2}$. To this end, let $b \in [a]_{R_1}$. Then $(a, b) \in R_1 \subseteq R_2$. Therefore $b \in [a]_{R_2}$, as desired.

   Conversely, suppose that P_1 is a refinement of P_2. Since $a \in [a]_{R_2}$, the definition of "refinement" forces $[a]_{R_1} \subseteq [a]_{R_2}$ for all $a \in A$. This means that for all $b \in A$ we have $(a, b) \in R_1 \to (a, b) \in R_2$; in other words, $R_1 \subseteq R_2$.

4. (#44 in page 580) Determine whether these posets are lattices.

(a) $(\{1, 3, 6, 9, 12\}, |)$

(b) $(\{1, 5, 25, 125\}, |)$

(c) $(Z, \geq)$

(d) $(P(S), \supseteq)$, where $P(S)$ is the power set of a set S

Solution: In each case, we need to decide whether every pair of elements has a least upper bound and a greatest lower bound.

(a) This is not a lattice, since the elements 6 and 9 have no upper bound.

(b) This is a lattice; in fact it is a linear order, since each element in the list divides the next one. The least upper bound of two numbers in the list is the larger, and the greatest lower bound is the smaller.

(c) Again, this is a lattice because it is a linear order.

(d) This is similar to Example 24, with the roles of subset and superset reversed. Here the g.l.b. of two subsets A and B is $A \cup B$, and their l.u.b. is $A \cap B$.

5. (# 34 in page 610) How many edges does a graph have if its degree sequence is $4, 3, 3, 2, 2$? Draw such a graph.

   Solution: The 4-wheel (see Figure 5) with one edge along the rim deleted is such a graph. It has $(4 + 3 + 3 + 2 + 2)/2 = 7$ edges.

6. (#40 in page 632) Show that if a connected simple graph G is the union of the graphs G_1 and G_2, then G_1 and G_2 have at least one common vertex.

   Solution: By definition of graph, both G_1 and G_2 are nonempty. If they have no common vertex, then there clearly can be no path from $v \in G_1$ to $v_2 \in G_2$. In that case G would not be connected, contradicting the hypothesis.

7. (#10 in page 644) Can someone cross all the bridges shown in this map exactly once and return to the starting point? (The map is shown in the text in page 644.)

   Solution: Each vertex has even degree, so the graph has an Euler ciruit. Therefore a walk of the type described is possible.

8. (#48 in page 646) Can you find a simple graph with n vertices with $n \geq 3$ that does not have a Hamilton circuit, yet the degree of every vertex in the graph is at least $(n - 1)/2$?

   Solution: We want to look only at odd n, since if n is even, then being at least $(n - 1)/2$ is the same as being at least $n/2$, in which case Dirac's Theorem would apply. One way to avoid having a Hamilton circuit is to have a cut vertex – a vertex whose removal disconnects the graph. The simplest example would be the "bow-tie" graph with five vertices (a, b, c, d, e), where cut vertex c is adjacent to each of the other vertices, and the only other edge are ab and de. Every vertex has degree at least $(5-1)/2 = 2$, but there is no Hamilton circuit.