Illinois Institute of Technology
Department of Computer Science

CS 330 Discrete Structures

Fall Semester, 2010

Homework Assignment 5

Due date: 4th Oct., 2010

1. (#66 in page 282) Suppose there are n people in a group, each aware of a scandal no one else in the group knows about. These people communicate by telephone; when two people in the group talk, they share information about all scandals each knows about. For example, on the first call, two people share information, so by the end of the call, each of these people knows about two scandals. The **gossip problem** asks for $G(n)$, the minimum number of telephone calls that are needed for all n people to learn about all the scandals.

   Use mathematical induction to prove that $G(n) \leq 2n - 4$ for $n \geq 4$. [Hint: In the inductive step, have a new person call a particular person at the start and at the end.]

   Solution: Basis: $G(4) = 4 \leq 2 \times 4 - 4$ Induction: Suppose $G(k) \leq 2k - 4$. When there are $k+1$ people, suppose the $k+1$-th people is x. x firt tell one of the other k people about the scandal he knows. Then the rest of them to share the scandals only need $G(k)$ times, then at the last any of them call back to x and all of them know all the scandals. So $G(k+1) = G(k) + 2 \leq 2k - 2 = 2(k+1) - 4$. Thus, proved.

2. (# 70 in page 282) Suppose that we want to prove that:

$$\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{3n}}$$

   for all positive integers n.

   (a) Show that if we try to prove this inequality using mathematical induction, the basis step works, but the inductive step fails.

   Solution: Basis: $\frac{1}{2} < \frac{1}{\sqrt{3}}$, correct. Inductive: Suppose $\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2k-1}{2k} < \frac{1}{\sqrt{3k}}$, then we want to prove $\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2k+1}{2k+2} < \frac{1}{\sqrt{3k+3}}$. By inductive hypothesis, $\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2k-1}{2k} \cdot \frac{2k+1}{2k+2} < \frac{1}{\sqrt{3k}} \cdot \frac{2k+1}{2k+2}$, we want this value smaller than $\frac{1}{\sqrt{3k+6}}$. However, by calculation, we can get that we require $\frac{3k+1}{(2k+1)^2} - \frac{1}{2k+1} - 2 > 0$, which is impossible because $\frac{3k+1}{(2k+1)^2} < 1$.

   (b) Show that mathematical induction can be used to prove the stronger inequality

$$\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{3n+1}}$$

   for all integers greater than 1, which , together with a verification for the case where $n = 1$, establishes the weaker inequality we originally tried to prove using mathematical induction.

   Solution: Basis is correct trivially. Inductive step is quite similar as what we did for previous one. And the inequality holds under this condition.

3. (#14 in page 292) Suppose you begin with a pile of n stones and split this pile into n piles of one stone each by successively splitting a pile of stones into two smaller piles. Each time you split a pile you

multiply the number of stones in each of the two smaller piles you form. so that if these piles have r and s stones in them, resp. you computer rs. Show that no matter how you split the piles, the sum of the products computed at each step equals $\frac{n(n-1)}{2}$.

Solution: It is clearly true for $n = 1$. Assume the strong inductive hypothesis, and suppose that our first splitting is into piles of i stones and $n - i$ stones, where i is a positive integer less than n. This gives a product $i(n-i)$. The rest of the products will be obtained from splitting the piles thus formed, and so by the inductive hypothesis, the sum of the products will be $i(i-1)/2 + (n-i)(n-i-1)/2$. So we want to show that $i(n-i) + i(i-1)/2 + (n-i)(n-i-1)/2 = \frac{n(n-1)}{2}$ no matter what i is. This follows by elementary algebra.

4. (# 18 in page 292) Use strong induction to show that when a convex polygon P with consecutive vertices $v_1, v_2, \cdots, v_n$ is triangulated into $n-2$ triangles, the $n-2$ triangles can be numbered $1, 2, \cdots, n-2$ so that v_i is a vertex of triangle i for $i = 1, 2, \cdots, n-2$.

   Solution: Basis: It is clearly true for a triangle. Inductive steps: We suppose that it is true for a $3, 4, \cdots, n$ sides polygon and prove it is also true for a $n+1$ sides polygon.

   The vertices are numbered with $1, 2, \cdots, n-1, n, n+1$ in the $n+1$ sides polygon. If we add one diagonal to the polygon such that we partition the polygon into a triangle $(n-1, n, n+1)$ and a n sides polygon $(1, 2, \cdots, n-2, n+1)$. And we number the triangle $(n-1, n, n+1)$ by $n-2$. So v_{n-2} is in the triangle $n-2$. By the induction hypothesis, we know that for the n sides polygon we can always find a partition such that each triangle i contains vertex v_i. So we are done.

5. (# 32 in page 293) Find the flaw with the following "proof" that every postage of three cents or more can be formed using just three-cent and four-cent stamps.

Basis step: We can form postage of three cents with a single three-cent stamp and we can form postage of four cents using a single four-cent stamp.

Inductive Step: Assume that we can form postage of j cent for all nonnegative integers j with $j \leq k$ using just three-cent and four-cent stamps. We can then form postage of $k+1$ cents by replacing one three-cent stamp with a four-cent stamp or by replacing two four-cent stamps by three three-cent stamps.

Solution: The proof is invalid for $k = 4$. That is the basis of the proof is invalid.

6. (# 26 in page 309) Let S be the subset of the set of ordered pairs of integers defined recursively by:

   Basis step: $(0, 0) \in S$.
   Recursive step: If $(a, b) \in S$, then $(a+2, b+3) \in S$ and $(a+3, b+2) \in S$.

   (a) List the elements of S produced by the first five applications of the recursive definition.
   Solution: $(0,0), (2,3), (3,2), (4,6), (5,5), (6,4), (6,9), (7,8), (8,7), (9,6), (10,15), (11,14), (12,13), (13,12), (14,11)$.

   (b) Use strong induction on the number of applications of the recursive step of the definition to show that $5 \mid a+b$ when $(a, b) \in S$.
   Solution: Let $P(n)$ denotes that after applying the recursive step for n steps, the resulted (a, b) satisfies $5 \mid (a+b)$. Basis: $P(0)$ is correct trivially. Inductive step: Suppose $\forall k \leq n-1, P(k)$ is correct. When $n = k+1$, we know that the sum of (a', b') obtained is $a+b+2+3 = a+b+5$, by applying the inductive hypothesis, we know that $5 \mid a' + b'$.

   (c) Use structural induction to show that $5 \mid a+b$ when $(a, b) \in S$.
   Solution: Because each time we apply the inductive step to get the next pair of numbers, the sum of them increases by 5. So after k iterations, the sum of them will be $5k$ which divides 5.

7. (# 44 in page 310) Use structural induction to show that $l(T)$, the number of leaves of a full binary tree T, is 1 more that $i(T)$, the number of internal vertices of T.

   Solution: Basis: Consider the tree only with a root, the relationship is clearly correct. Inductive step: Suppose that T is made up of T_1 and T_2 plus a new root and the relationship holds in T_1, T_2, then the number of internal node of T is $i(T) = 1 + i(T_1) + i(T_2)$, the number of leaves is $l(T) = l(T_1) + l(T_2) = i(T_1) + 1 + i(T_2) + 1 = i(T) + 1$. Thus we finished.