Illinois Institute of Technology
Department of Computer Science

CS 330 Discrete Structures

Fall Semester, 2010

Homework Assignment 4

Due date: 27th September, 2010

1. (#34 in page 147) Let $f(x) = ax + b$ and $g(x) = cx + d$, where a, b, c, and d are constants. Determine for which constants a, b, c, and d it is true that $f \circ g = g \circ f$.

   Solution: Forming the compositions we have $(f \circ g)(x) = acx + ad + b$ and $(g \circ f)(x) = cax + cb + d$. These are equal if and only if $ad + b = cb + d$. In other words, equiality holds for all 4-tuples (a, b, c, d) for which $ad + b = cb + d$.

2. (# 68 in page 148) Suppose that f is a function fromm A to B, where A and B are finite sets with $|A| = |B|$. Show that f is one-to-one if and only if it is onto.

   Solution: If f is one-to-one, then every element of A gets sent to a different element of B. If in addition to the range of A there were another element in B, then $|B|$ would be at least one greater than $|A|$. This cannot happen, so we conclude that f is onto. Conversely, suppose that f is onto, so that every element of B is the image of some element of A. In particular, there is an element of A for each element of B. If two or more elements of A were sent to the same elements of B, then $|A|$ would be at least one greater than the $|B|$. This connot happen, so we conclude that f is onto.

3. (# 20 in page 191) Give a big_O estimate for each of these functions. For the function g in y our estimate $f(x)$ is $O(g)$, use a simple function g of smallest order.

   (a) $(n^3 + n^2 \log n)(\log n + 1) + (17 \log n + 19)(n^3 + 2)$
   (b) $(2^n + n^2)(n^3 + 3^n)$
   (c) $(n^n + n2^n + 5^n)(n! + 5^n)$

   Solution:

   (a) $O(n^3 \cdot \log n)$
   (b) $O(6^n)$
   (c) $O(n^n \cdot n!)$

4. (# 40 in page 192) Show that if $f_1(x)$ and $f_2(x)$ are functions from the set of positive integers to the set of real numbers and $f_1(x)$ is $\Theta(g_1(x))$ and $f_2(x)$ is $\Theta(g_2(x))$, then $(f_1 f_2)(x)$ is $\Theta(g_1 g_2(x))$.

   Solution: The definitions tell us that there are positive constants C_1, k_1, C_2, k_2 such that $|f_1(x)| \leq C_2|g_1(x)|$ for all $x > k_2$ and $|f_1(x)| \geq C_1|g_1(x)|$ for all $x > k_1$. Similarly, we have that there are positive constants C_1', k_1', C_2', k_2' such that $|f_2(x)| \leq C_2'|g_2(x)|$ for all $x > k_2'$ and $|f_2(x)| \geq C_1'|g_2(x)|$ for all $x > k_1'$. We can combine these ineqalities to obtain $|f_1(x)f_2(x)| \leq C_2C_2'|g_1(x)g_2(x)|$ for all $x > \max\{k_2, k_2'\}$ and $|f_1(x)f_2(x)| \geq C_1C_1'|g_1(x)g_2(x)|$ for all $x > \max\{k_1, k_1'\}$. This means that $f_1(x)f_2(x)$ is $\Theta(g_1(x)g_2(x))$.

5. (# 52 in page 192) *(Requires calculus)*

(a) Show that if $f(x)$ and $g(x)$ are functions such that $f(x)$ is $o(g(x))$ and c is a constant, then $cf(x)$ is $o(g(x))$, where $(cf)(x) = cf(x)$.

(b) Show that if $f_1(x)$, $f_2(x)$, and $g(x)$ are functions such that $f_1(x)$ is $o(g(x))$ and $f_2(x)$ is $o(g(x))$, then $(f_1 + f_2)(x)$ is $o(g(x))$, where $(f_1 + f_2)(x) = f_1(x) + f_2(x)$.

Solution:

(a) $\lim_{x\to\infty} \frac{cf(x)}{g(x)} = c \lim_{x\to\infty} \frac{f(x)}{g(x)} = c \cdot 0 = 0$.

(b) $\lim_{x\to\infty} \frac{(f_1+f_2)(x)}{g(x)} = \lim_{x\to\infty} \frac{f_1(x)}{g(x)} + \lim_{x\to\infty} \frac{f_2(x)}{g(x)} = 0 + 0 = 0$.

6. (# 22 in page 209) Show that if a, b, c, and m are integers such that $m \geq 2, c > 0$, and $a \equiv b(\text{mod} m)$, then $ac \equiv bc(\text{mod} mc)$.

Solution: From $a \equiv b(\text{mod} m)$ we know that $b = a + sm$ for some integer s. Multiplyig by c we have $bc = ac + s(mc)$, which means that $ac \equiv bc(\text{mod} mc)$.

7. (#42 in page 281) Prove that if $A_1, A_2, \ldots, A_n$ and B are sets, then

$$(A_1 - B) \cap (A_2 - B) \cap \ldots \cap (A_n - B) = (A_1 \cap A_2 \cap \ldots \cap A_n) - B.$$

Solution: If $n = 1$ there is nothing to prove, and the $n = 2$ case says that $(A_1 \cap \bar{B}) \cap (A_2 \cap \bar{B}) = (A_1 \cap A_2) \cap \bar{B}$, which is certainly true, since an element is in each side if and only if it is in all three of the sets $A_1, A_2, \bar{B}$. Those take care of the basis step. For the inductive step, assume that

$$(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) = (A_1 \cap A_2 \cap \cdots \cap A_n) - B$$

; we must show that

$$(A_1 - B) \cap \cdots \cap (A_{n+1} - B) = (A_1 \cap \cdots \cap A_{n+1}) - B$$

We have

$$(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) \cap (A_{n+1} - B) = ((A_1 \cap A_2 \cap \cdots \cap A_n) - B) \cap (A_{n+1} - B) = (A_1 \cap \cdots \cap A_{n+1}) - B$$

The first equality is from the inductive hypothesis, and the final equality is from the $n = 2$ case.