Illinois Institute of Technology
Department of Computer Science

CS 330 Discrete Structures

Fall Semester, 2010

MidTerm

Nov. 8th, 2010

1. In this exercise we will derive a formula for the sum of the squares of the n smallest positive integers. We will count the number of triples (i, j, k) such that i, j, k are integers such that $0 \leq i < k, 0 \leq j < k$, and $1 \leq k \leq n$ in two ways.

   (a) Show that there are k^2 such triples with a fixed k. Conclude that there are $\sum_{k=1}^{n} k^2$ such triples.

   Solution: With a fixed k, either i or j can be chosen from k numbers, from $\{0, 1, \cdots, k-1\}$. So there will be k^2 such triples. Considering that k can be chosen from 1 to n, so there are $\sum_{k=1}^{n} k^2$ such triples.

   (b) Show that the number of such triples with $0 \leq i < j < k$ and the number of such triples with $0 \leq j < i < k$ both equal $C(n+1, 3)$.

   Solution: Please note that in both cases, i, j, k are always different with fixed order. So for any 3 numbers, there is only one way to assign i, j, k. We can choose the numbers from the range of $\{0, 1, \cdots, n\}$, $n+1$ numbers. So there are $C(n+1, 3)$ ways to choose.

   (c) Show that the number of such triples with $0 \leq i = j < k$ equals $C(n+1, 2)$.

   Solution: When $i = j$, we only need to choose 2 numbers from $n+1$, the smaller one is assigned to i, j, and the larger one is assigned to k. So there are $C(n+1, 2)$ ways to choose the triples.

   (d) Combining part (a) with parts (b) and (c), conclude that

   $$\sum_{k=1}^{n} k^2 = 2C(n+1, 3) + C(n+1, 2) = \frac{n(n+1)(2n+1)}{6}.$$

   Solution: To calculate the total number of triples, we need to sum all the cases in (b), and (c). Please note that there are two symmetry cases in (b), so the total number of triples shoulda be $2C(n+1, 3) + C(n+1, 2)$. From (a) we also know that the total number of triples is $\sum_{k=1}^{n} k^2$. So $\sum_{k=1}^{n} k^2 = 2C(n+1, 3) + C(n+1, 2)$.

   $$2C(n+1, 3) + C(n+1, 2) = 2\frac{(n+1)n(n-1)}{3 \cdot 2} + \frac{(n+1)n}{2} = \frac{n(n+1)(2n+1)}{6}.$$

2. How many ways are there to seat six boys and eight girls in a row of chairs so that no two boys are seated next to each other?

   Solution: We first permutate the eight girls, which is 8! ways. After we permutated the girls, there will be 9 vacants to seat the six boys, because no two boys are seated next to each other. So there are $P(9, 6)$ ways to seat the boys. Put them together, there are $8! \cdot P(9, 6)$ ways.

3. Use mathematical induction to show that if you draw lines in the plane you only need two colors to color the regions formed so that no two regions that have an edge in common have a common color.

Solution: Basis step: If there is only one line, then the whole region is divided into 2 parts, we color the region to the right of (or above) the line with color 1, and color the region to the left of (or below) the line with color 2. So it is correct.

Inductive step: Suppose that we can color the regions partitioned by the k lines with 2 colors. Then we add the $k+1$th line. It will partition the whole area into two parts (left and right, or above and below). For the regions to the right of the line, we keep their colors as previous. For the regions to the left of the line, we flip their colors. By this way, it is clear to see that the regions not partitioned by the $(k+1)$th line satisfy the property. For the regions partitioned by the $(k+1)$th line, the two regions share a common edge have different colors. So it is correct with $k+1$ lines.

Thus we proved it.

4. The set B of all **balanced strings of parentheses** is defined recursively by $\lambda \in B$, where λ is the empty string; $(x) \in B$, $xy \in B$ if $x, y \in B$. Use induction to show that if x is a balanced string of parentheses, then the number of left parentheses equals the number of right parentheses in x.

   Solution: Basis step: For the empty string λ, it is clear that the number of left parentheses equals to the number of right parentheses, and they both equal to 0.

   Inductive step: We just need to prove that given strings of parentheses which are balanced, the new strings generated by the rules given will always be balanced. Suppose in x the number of left parentheses equals the number of right parentheses. By the first rule $x \in B \rightarrow (x) \in B$. It is clear that in (x), the left parentheses equals the number of right parentheses. Suppose in either x or y the number of left parentheses equals the number of right parentheses, then by the second rule $xy \in B$. It is clear that xy is also balanced.

   Thus we proved for any balanced strings of parentheses, the number of left parentheses equals the number of right parentheses.