Illinois Institute of Technology
Department of Computer Science

CS 330 Discrete Structures

Fall Semester, 2010

Homework Assignment 9

Due date: 24th Nov., 2010

1. (#18 in page 457)

   (a) Find a recurrence relation for the number of permutations of a set with n elements.

   Solution: A permutation of a set with n elements consists of a choice of a first element, followed by a permutation of a set with $n-1$ elements. Therefore, $P_n = nP_{n-1}$. Note that $P_0 = 1$, since there is just one permutation of a set with no objects, namely the empty sequence.

   (b) Use this recurrence relation to find the number of permutations of a set with n elements using iteration.

   Solution: $P_n = nP_{n-1} = n(n-1)P_{n-2} = \cdots = n(n-1)\cdots 2 \cdot 1 \cdot P_0 = n!$

2. (#24 in page 458)

   (a) Find a recurrence relation for the number of bit strings of length n that contain three consecutive 0s.

   Solution: Let a_n be the number of bit strings of length n containing three consecutive 0's. In order to construct a bit strig of length n containing three consecutive 0's we could start with 1 and follow with a string of length $n-1$ containing three consecutive 0's, or we could start with a 01 and follow with a string of length $n-2$ containing three consecutive 0's, or we could start with a 001 and follow with a string of length $n-3$, or we could start with a 000 and follow with any string of length $n-3$. These four cases are mutually exclusive and exhaust the possibilities for how the string might start. From this analysis we can immediately write down the recurrence relation, valid for all $n \geq 3 : a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3}$.

   (b) What are the initial conditions?

   Solution: There are no bit strings of length $0, 1, 2$ containing three consecutive 0's, so the initial conditions are $a_0 = a_1 = a_2 = 0$.

   (c) How many bit strings of length seven contain three consecutive 0s.

   Solution: We will compute a_3 through a_7 using the recurrence relation:

   $$a_3 = 1, a_4 = 3, a_5 = 8, a_6 = 20, a_7 = 47.$$

   Thus thee are 47 bit strings of length 7 containing three consecutive 0's.

   [Note:] For the question to calculate how many bit strings of length 7 do not contain three consecutive 0's, you can find its answer in the back of the textbook, which is calculated by the recurrence relation derived similarly as we did here but with different initial conditions.

3. (#28 in page 472)

(a) Find all solutions of the recurrence relation $a_n = 2a_{n-1} + 2n^2$.

Solution: The associated homogeneous recurrence relation is $a_n = 2a_{n-1}$. We easily solve it to obtain $a_n^{(h)} = \alpha 2^n$. Next we need a particular solution to the given recurrence relation. By Theorem 6 we want to look for a function of the form $a_n = p_2n^2 + p_1n + p_0$. We plug this into our recurrence relation and obtain $p_2n^2+p_1n+p_0 = 2(p_2(n-1)^2+p_1(n-1)+p_0)+2n^2$. We rewrite this by grouping terms with equal powers of n, obtaining $(-p_2-2)n^2+(4p_2-p_1)n+(-2p_2+2p_1-p_0) = 0$. In order for this equation to be true for all n, we must have $p_2 = -2, 4p_2 = p_1, -2p_2 + 2p_1 - p_0 = 0$. This tells us that $p_1 = -8, p_0 = -12$. Therefore the particular solution we seek is $a_n^{(p)} = -2n^2 - 8n - 12$. So the general solution is the sum of the homogeneous solution and this particular solution, namely $a_n = \alpha 2^n - 2n^2 - 8n - 12$.

(b) Find the solution of the recurrence relation in part (a) with initial condition $a_1 = 4$.

Solution: We plug the initial condition into our solution from part (a) to obtain $4 = a_1 = 2\alpha - 2 - 8 - 12$. This tells us that $\alpha = 13$. So the solution is $a_n = 13 \cdot 2^n - 2n^2 - 8n - 12$.

4. (#34 in page 472) Find all solutions of the recurrence relation $a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3} + n4^n$ with $a_0 = -2, a_1 = 0$, and $a_2 = 5$.

Solution: The associated homogeneous recurrence relation is $a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3}$. To solve it we find the characteristic equation $r^3 - 7r^2 + 16r - 12 = 0$. By the retional root test we soon discover that $r = 2$ is a root and factor our equation into $(r-2)^2(r-3) = 0$. Therefore the general solution of the homogeneous relation is $a_n^{(h)} = \alpha 2^n + \beta n \cdot 2^n + \gamma 3^n$. Next we need a particular solution to the given recurrence relation. By Theorem 6 we want to look for a function of the form $a_n = (cn + d)4^n$, since the coefficient of 4^n in our given relation is a linear function of n, and 4 is not a root of the characteristic equation. We plug this into our recurrence relation and obtain $(cn + d)4^n = 7(cn-c+d)4^{n-1}-16(cn-2c+d)4^{n-2}+12(cn-3c+d)4^{n-3}+n4^n$. We divide through $c = 16$ and $d = -80$, so the particular solution we seek is $a_n^{(p)} = (16n - 80)4^n$. Thus the general solution is the sum of the homogeneous solution and this particular solution, namely $a_n = \alpha 2^n+\beta n\cdot 2^n+\gamma 3^n+(16n-80)4^n$. Next we plug in the initial conditions and obtain $-2 = a_0 = \alpha + \gamma - 80, 0 = a_1 = 2\alpha, 2\beta + 3\gamma - 256, 5 = a_2 = 4\alpha+8\beta+9\gamma-768$. We solve this system of three linear equations in three unknowns by standard methods to obtain $\alpha = 17, \beta = 39/2, \gamma = 61$. So the solution is $a_n = 17 \cdot 2^n + 39n2^{n-1} + 61 \cdot 3^n + (16n - 80)4^n$. As a check of our work, we can compute a_3 both from the recurrence and from the solution, and we find that $a_3 = 203$ both ways.

5. (# 12 in page 482) Find $f(n)$ when $n = 3^k$, where f satisfies the recurrence relation $f(n) = 2f(n/3)+4$ with $f(1) = 1$.

Solution: An exact formula comes from the proof of Theorem 1, namely $f(n) = [f(1)+c/(a-1)]n^{\log_b a} - c/(a-1)$, where $a = 2, b = 3, c = 4$ in this exercise. Therefore the answer is $f(n) = 5n^{\log_3 2} - 4$.

6. (#38 in page 498) Use generating functions to solve the recurrence relation $a_k = 2a_{k-1}+3a_{k-2}+4^k+6$ with initial conditions $a_0 = 20, a_1 = 60$.

Solution: Let $G(x)$ be the generating function of the sequence $a_0, a_1, \cdots$. If we sum the equation $a_k = 2a_{k-1} + 3a_{k-2} + 4^k + 6$ from $k = 2$ to ∞, then we got:

$$G(x) - 20 - 60x = 2xG(x) + 3x^2G(x) - 40x + 16x^2/(1-4x) + 6x^2/(1-x).$$

We can calculate $G(x) = \frac{-20+80x-2x^2-40x^3}{(x+1)(3x-1)(1-4x)(1-x)} = \frac{A}{x+1} + \frac{B}{3x-1} + \frac{C}{1-4x} + \frac{D}{1-x}$. And we can calculate that $A = \frac{31}{20}, B = -\frac{67}{4}, C = -\frac{2}{5}, D = -\frac{2}{3}$. So $G(x) = \frac{31/20}{1+x} + \frac{-67/4}{3x-1} + \frac{-2/5}{1-4x} + \frac{-2/3}{1-x}$. From the table, we can get that $a_k = \frac{31}{20}(-1)^n + \frac{67}{4} \cdot 3^n - \frac{2}{5}4^n - \frac{2}{3}$.

7. (#55 in page 529) Let R be a reflexive relation on a set A. Show that R^n is reflexive for all positive integer n.

   Solution: We can just use mathematical induction to prove it. Please see the page S-47 in the textbook.