Illinois Institute of Technology
Department of Computer Science

CS 330 Discrete Structures

Fall Semester, 2010

Homework Assignment 7

Due date: 1st Nov., 2010

1. (#24 in page 361 How many ways are there for 10 women and six men to stand in a line so that no two men stand next to each other ? [Hint: First position the women and then consider possible positions for the men.]

   Solution: First position the women relative to each other. Since there are 10 women, there are $P(10, 10)$ ways to do this. This creates 11 slots whee a man (but not more than one man) may standl: in front of the first woman, between the first and second women, ..., between the ninth and the tenth women, and behind the tenth woman. We need to choose six of these positions, in order, for the first through six man to occupy. This can be done in $P(11, 6)$ ways. Therefore the answer is $P(10, 10) \cdot P(11, 6)$.

2. (#42 in page 362) How many ways are there for a horse race with four horses to finish if ties are possible?(Note: two or three horses may tie. Any number of the four horses may tie.)

   Solution: We can solve this problem by breaking it down into cases depending on the number of ties. There are five cases. (1) If there are no ties, then there are clearly $P(4, 4)$ possible ways for the horses to finish. (2) Assume that there are two horses that tie, but the others have distinct finishes. There are $C(4, 2)$ wys to choose the horses to be tied; then there are $P(3, 3)$ ways to determine the order of finish for the three groups. Thus there are $C(4, 2)P(3, 3)$ ways. (3) There are two groups of two horses that are tied. There are $C(4, 2)$ ways to choose the winners. (4) There might be a group of three horses all tied. There are $C(4, 3)$ ways to choosee which these horses will be, and then two ways for the race to end. (5) There is only one way for all the horses to tie. Putting this all together, the answer is $24 + 36 + 6 + 8 + 1 = 75$.

3. (#24 in page 369) Show that if p is a prime and k is an integer such that $1 \leq k \leq p - 1$, then p divides $\begin{pmatrix} \mathrm{p} \\ \mathrm{k} \end{pmatrix}$.

   Solution: $C(p, k) = \frac{p!}{k! \cdot (p-k)!} = \frac{p}{k} \cdot C(p - 1, k - 1)$. So $k \cdot C(p, k) = p \cdot C(p - 1, k - 1)$. Since $C(p, k)$ is an integer, and p cannot divide k, if p divides $k \cdot C(p, k)$, then p divides $C(p, k)$. Clearly, p divides $k \cdot C(p, k)$. So wa are done.

4. (#28 in page 369) Show that if n is a positive integer, then $\begin{pmatrix} 2\mathrm{n} \\ 2 \end{pmatrix} = 2 \begin{pmatrix} \mathrm{n} \\ 2 \end{pmatrix} + n^2$.

   (a) using a combinatorial argument.
   Solution: To choose 2 people from a set of n men and n women, we can either choose 2 men or 2 women or one of each sex. Therefore the right-hand side counts the number of ways to do this by sum rule. The left-hand side counts the same thing, since we are simply choosing 2 people from $2n$ people.

   (b) using Pascal's identity.
   $2C(n, 2) + n^2 = n(n - 1) + n^2 = 2n^2 - n = n(2n - 1) = 2n(2n - 1)/2 = C(2n, 2)$.

5. (# 24 in page 380) How many ways are there to distribute 15 distinguishable objects in to five distinguishable boxes so that the boxes have one, two, three, four and five objects in them, respectively.

   Solution: We assume that this problem leaves us free to pick which boxes get which number of balls. There are several ways to count this. Here is one. Line up the 15 objects in a row (15! ways), and line up the five boxes in a row (5! ways). Now put the first object into the first box, the next two into the second box, the next three into the third box, and so on. This overcounts by a factor of $1! \cdot 2! \cdot 3! \cdot 4! \cdot 5!$, since there are that many ways to swap objects in the permutation without affecting the result. Therefore the answer is $\frac{15! \cdot 5!}{1! \cdot 2! \cdot 3! \cdot 4! \cdot 5!}$ ways.

6. (#44 in page 381) In how many wats can a dozen books be placed on four distinguishable shelves

   (a) if the books are indistinguishable copies of the same title?

   Solution: All that matters is the number of books on each shelf, so the answer is the number of solutions to $x_1 + x_2 + x_3 + x_4 = 12$. The answer is therefore $C(4 + 12 - 1, 12)$.

   (b) if no two books are same, and the positions of the books on the shelves matter? [Hint: Break this into 12 tasks, placing each book separately. Start with the sequence 1,2,3,4 to represent the shelves. Represent the books by $b_i, i = 1, 2, \cdots, 12$. Place b_1 to the right of one of the terms in 1,2,3,4. Then successively place $b_2, b_3, \cdots, b_12$.]

   Solution: No generality is lost if we number the books $b_1, b_2, b_3, b_4, \cdots, b_{12}$ and think of placing book b_1, then placing b_2, and so on. There are clearly 4 ways to place b_1, since we can put it as the first book on any of shelves. After b_1 placed, there are 5 ways to place b_2, and so on. So totally, there are $\frac{15!}{(15-12)!}$ ways.

7. (#58 in page 381) How many ways are there to distribute five balls into seven boxes if each box must have at most one ball in it if

   (a) both the balls and boxes are labeled?

   Solution: There are 7 choices for the first ball, 6 choices for the second ball, and so on. So $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3$.

   (b) the balls are labeled, but the boxes are unlabeled?

   Solution: Since each ball must be in a separate box and the boxes are unlabeled, there is only one way to do this.

   (c) the balls are unlabeled, but the boxes are labeled?

   Solution: This is just a matter of choosing which five boxes to put balls into, so the answer is $C(7, 5)$.

   (d) both the balls and boxes are unlabled?

   Solution: As noted in (b), there is only one way to do this.