Illinois Institute of Technology
Department of Computer Science

CS 330 Discrete Structures

Fall Semester, 2010

Homework Assignment 1 – Solution

Due date: 1st September, 2010

The questions weight the same and each of them weights 10 points. So the total points is the total # of questions times 10.

1. In Lopsided Trees if the costs are α (an integer) and 1 then what are the range of weights possible at the leaves of an optimal tree? Can you write an expression (a recurrence relation- as done in class) for the maximum number of nodes of cost w? (You dont have to solve the recurrence)

   Solution: If the most weighted leaf in the Lopsided Tree is w, then we can get that the least weighted leaf in this tree is $\geq w - \alpha$, which can be proved below:

   Suppose that there is some leaf L_0 with weight $w' < w - \alpha$ in the optimal Lopsided Tree. Since we know that the most weighted leaf L_1 in the tree is w and it must have a sibling L_2 with weight $w - \alpha + 1$ and have a parent L_P with weight $w - \alpha$. Then we can move L_1 and L_2 from L_P to L_0 resulting a smaller weighted tree; a new leaf L_P with weight $w - \alpha$ and two new leaves L_1 and L_2 with weights $w' + \alpha < w$ and $w' + 1 < w$ respectively. The resulting tree has a smaller weight which contradicts the assumption that the original tree is optimal. So the least weighted leaf could be at least $w - \alpha$.

   So the possible range of the weight in an optimal tree is $[w - \alpha, w]$ if the most weighted leaf weights w.

   The expression for the maximum number of nodes of cost w is:

$$M_w = M_{w-\alpha} + M_{w-1}$$

2. (#46 in page 20) An explorer is captured by a group of cannibals. There are two types of cannibals – those who always tell the truth and those who always lie. The cannibals will barbecue the explorer unless he can determine whether a particular cannibal always lies or always tells the truth. He is allowed to ask the cannibal exactly one question.

   (a) Explain why the question "Are you a liar?" does not work.

   Solution: The cannibals that always tell the truth will answer "No", while the others always lie will answer also "No". Thus we cannot determine.

   (b) Find a question that the explorer can use to determine whether the cannibal always lies or always tells the truth.

   Solution: Just tell something is always true (or always false), such as "Is $1 + 1 = 2$?", or "Does sun rise from the east?", or "Are you both liers?" Thus we can clearly determine which one tells the truth and while one lies.

3. (#56 in page 20) This exercise relates to inhabitants of the island of knights and knaves created by Smullyan, where knights always tell the truth and knaves always lie. You encounter two people, A and

B. Determine, if possible, what A and B are if they address you in the way described. If you cannot determine what these two people are, can you draw any conclusions?

A says "The two of us are both knights" and B says "A is a knave."

Solution: Let p represents A is a knight and q represents B is a knight. The above statement can be represented as

$$(p \leftrightarrow (p \wedge q)) \wedge (q \leftrightarrow \neg p)$$

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Let's draw a truth table as follows:

p	q	$(p \leftarrow (p \wedge q)) \wedge (p \rightarrow (p \wedge q)) \wedge (q \leftarrow \neg p) \wedge (q \rightarrow \neg p)$
T	T	F
T	F	F
F	T	T
F	F	F

From the truth table we can see that the statement $(p \leftrightarrow (p \wedge q)) \wedge (q \leftrightarrow \neg p) = T$ iff $p = F \wedge q = T$. So we got A is a knave and B is a knight.

4. (#62 in page 21) Five friends have access to a chat room. Is it possible to determine who is chatting if the following information is known?

   (a) Either Kevin or Heather, or both, are chatting.

   (b) Either Randy or Vijay, but not both, are chatting.

   (c) If Abby is chatting, so is Randy.

   (d) Vijay and Kevin are either both chatting or neither is.

   (e) If Heather is chatting, then so are Abby and Kevin.

   Explain your reasoning.

   Solution: If we use the name's initial represents that guy is chatting, then we got

   1 $K \vee H = T$

   2 $(\neg R \wedge V) \vee (R \wedge \neg V) = T$

   3 $A \rightarrow R = T$

   4 $(V \wedge K) \vee (\neg V \wedge \neg K) = T$

   5 $H \rightarrow A \wedge K = T$

   From the statements above, we suppose $H = T$, then $A = T \wedge K = T$ by (5). Then $R = T$ by (3). Then $V = F$ by (2). And (1) and (4) both correct. So we got a correct answer:$H = A = K = R = \neg V = T$, that is only Vijay is not chatting.

   NOTE: Here is only one solution, some other solution may also be possible. One solution will get the full points of this question. To get all the correct solutions, you may need to check all the possible conditions based on truth table, however, it is not required here.

5. (#64 in page 21) Four friends have been identified as suspects for an unauthorized access into a computer system. They have made statements to the investigating authorities. Alice said "Carlos did it." John said "I did not do it." Carlos said "Diana did it." Diana said " Carlos lied when he said that I did it."

(a) If the authorities also know that exactly one of the four suspects is telling the truth, who did it? Explain your reasoning.

(b) If the authorities also know that exactly one is lying, who did it? Explain your reasoning.

Solution: From the statements of the four suspects, we can get that one of Carlos and Diana must lie, because the statements of Carlos and Diana is contradictory. Furthermore, if Diana lied then Carlos must tell the truth and if Carlos lied then Diana must tell the truth.

Let $N(X)$ represents X is telling the truth and $D(X)$ represents X did it.And let the name's initial represent that guy. Then

1 $N(A) \leftrightarrow D(C)$

2 $N(J) \leftrightarrow \neg D(J)$

3 $N(C) \leftrightarrow D(D)$

4 $N(D) \leftrightarrow D(D) \rightarrow \neg N(C)$

Case 1 – There is only one telling the truth: Suppose $N(C)$. Then $\neg D(C) \wedge D(J) \wedge D(D) \wedge \neg(D(D) \rightarrow \neg N(C))$. So $D(J) \wedge D(D)$, that is John and Diana did it. Suppose $N(D)$. Then $\neg D(C) \wedge D(J) \wedge \neg D(D) \wedge D(D) \rightarrow \neg N(C)$. So $D(J)$, that is John did it.

Case 2 – There is only one lying. Suppose $\neg N(C)$. Then $D(C) \wedge \neg D(J) \wedge \neg D(D) \wedge D(D) \rightarrow \neg N(C)$. So $D(C)$, that is Carlos did it. Suppose $\neg N(D)$. Then $D(C) \wedge \neg D(J) \wedge D(D) \wedge \neg(D(D) \rightarrow \neg N(C))$. So $D(C) \wedge D(D)$, that is Carlos and Diana did it.