Illinois Institute of Technology
Department of Computer Science

CS 330 Discrete Structures

Fall Semester, 2010

Homework Assignment 8

Due date: 15th Nov., 2010

1. (#32 in page 399 Suppose that 100 people enter a contest and that different winners are selected at random for first, second, and third prizes. What is the probability that Kumar, Janice, and Pedro each win a prize if each has entered the contest?

   Solution: The number of ways for the drawing to turn out is $100 \cdot 99 \cdot 98$. The number of ways of ways for the drawing to cause Kumar, Janice, and Pedro each to win a prize is $3 \cdot 2 \cdot 1$ (three ways for one of these to be picked to win first prize, two ways for one of the others to win second prize, one ways for the third to win the third prize). Therefore the probability we seek is $\frac{3 \cdot 2 \cdot 1}{100 \cdot 99 \cdot 98} = 1/161700$.

2. (#36 in page 399) Which is more likely: rolling a total of 8 when two dice are rolled or rolling a total of 8 when three dice are rolled?

   Solution: There are 5 ways to get a total of 8 when two dice are rolled: $(6,2),(5,3),(4,4),(3,5),(2,6)$. There are totally 36 possibilities. So the probability is $5/36$. There are 21 ways when there are three dice. And there are totally $6^3 = 216$ possibilities. So the probability is $21/216$. Thus rolling a total of 8 is more likely when using two dice than when using three.

3. (#18 in page 415)

   (a) What is the probability that two people chosen at random were born on the same day of the week?

   Solution: The probability that the second person has the same birth day-of-the-week as the first person is $1/7$.

   (b) What is the probability that in a group of n people chosen at random, there are at least two born on the same day of the week?

   Solution: We proceed as in Example 13. The probability that all the birth days-of-the-week are different is $\frac{6}{7}\frac{5}{7}\frac{4}{7}\cdots\frac{8-n}{7}$. When $n \geq 8$ the probability will always be 0.

   (c) How many people chosen at random are needed to make the probability greater than $\frac{1}{2}$ that there are at least two people born on the same day of the week?

   Solution: We compute $1 - p_n$ for $n = 2, 3, \cdots$ and find that the first time this exceeds $1/2$ is when $n = 4$, so that is our answer. With four people, the probability that at least two will share a birth day-of-the-week is $233/343$, or about 65%.

4. (#24 in page 415) What is the conditional probability that exactly four heads appear when a fair coin is flipped five times, given that the first flip came up tails?

   Solution: There are 16 equally likely outcomes of flipping a fair coin five times in which the first flip comes up tails. Of these only one will result in four heads appearing namely THHHH. Therefore the answer is $1/16$.

5. (# 34 in page 416) Find each of the following probabilities when n independent Bernoulli trials are carried out with probability of success p

(a) the probability of no successes.

Solution:$(1-p)^n$.

(b) the probability of at least one success

Solution: $1-(1-p)^n$.

(c) the probability of at most one success

Solution: For exactly one success, the probability is $n \cdot p \cdot (1-p)^{n-1}$. So the answer is $(1-p)^n + n \cdot p \cdot (1-p)^{n-1}$. Note that the answer only makes sense if $n > 0$; the probability is clearly 1 if $n = 0$.

(d) the probability of at least two successes

Solution: The answer is just the one minus the probability of (c).

6. (#16 in page 440) Let X and Y be the random variables that count the number of heads and the number of tails that come up when two coins are flipped. Show that X and Y are not independent.

   Solution: We need to show that $P(X = i \wedge Y = j)$ is not always equal to $P(X = i)P(Y = j)$. If we try $i = j = 2$, then the former will be 0 (since the sum of the number of heads and the number of the tails has to be 2), whereas the latter is $(1/4)(1/4) = 1/16$.

7. (#22 in page 440) Let $X(s)$ be a random variable, where $X(s)$ is a nonnegative integer for all $s \in S$, and let A_k be the event that $X(s) \geq k$. Show that $E(X) = \sum_{k=1}^{\infty} p(A_k)$.

   Solution: By definition, $E(X) = \sum_{k=1}^{\infty} k \cdot P(X = k)$. Let us write this out and regroup: $E(X) = P(X = 1)+(P(X = 2)+P(X = 2))+\cdot = (P(X = 1)+P(X = 2)+\cdots)+(P(X = 2)+P(X = 3)+\cdots)+\cdots$. But this is precisely $P(A_1) + P(A_2) + \cdots$.