Disjoint Bases in a Polymatroid

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Abstract

Let $f : 2^N \rightarrow \mathcal{Z}^+$ be a polymatroid (an integer-valued non-decreasing submodular set function with $f(\emptyset) = 0$). A base of N is a subset $S \subseteq N$ such that $f(S) = f(N)$. We consider the problem of finding a maximum number of disjoint bases; we denote by m^* be this base packing number. A simple upper bound on m^* is given by $k^* = \max\{k : \sum_{i \in N} f_A(i) \geq k f_A(N)\}, \forall A \subseteq N\}$ where $f_A(S) = f(A \cup S) - f(A)$. This upper bound is a natural generalization of the bound for matroids where it is known that $m^* = k^*$. For polymatroids, we prove that $m^* \geq (1 - o(1))k^*/\ln f(N)$ and give a randomized polynomial time algorithm to find $(1-o(1))k^*/\ln f(N)$ disjoint bases, assuming an oracle for f. We also derandomize the algorithm using minwise independent permutations and give a deterministic algorithm that finds $(1 - \epsilon)k^*/\ln f(N)$ disjoint bases. The bound we obtain is almost tight since it is known there are polymatroids for which $m^* \leq (1 + o(1))k^*/\ln f(N)$. Moreover it is known that unless $NP \subseteq DTIME(n^{\log\log n})$, for any $\epsilon > 0$, there is no polynomial time algorithm to obtain a $(1 + \epsilon)/\ln f(N)$-approximation to m^*. Our result generalizes and unifies two results in the literature.

We indicate how a known connection between fractional packings and optimization implies hardness results for integer packings in a straightforward way. In particular, we show that the hardness of $(1 + \epsilon)/\ln f(N)$-approximation for the base packing number follows from the inapproximability of computing a minimum weight base in a polymatroid. We also prove that there is a constant $c > 0$ such that it is NP-hard to approximate the parameter k^* to within an $(c \log f(N))$-factor.

1 Introduction

A polymatroid $f : 2^N \rightarrow \mathcal{Z}^+$ on a ground set N is a non-decreasing (monotone) integer-valued submodular function. The function f is submodular iff $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$ for all $A, B \subseteq N$. The function is monotone if $f(A) \leq f(B)$ for all $A \subseteq B$. For a subset $A \subseteq N$, the function $f_A : 2^N \rightarrow \mathcal{Z}^+$ is defined by $f_A(S) = f(S \cup A) - f(S)$. It is easy to check that f_A is submodular if f is submodular. Polymatroids generalize matroids which have the additional condition that $f(i) \leq 1$ for all $i \in N$. We call a subset $S \subseteq N$ a *base*[1] if $f(S) = f(N)$. We consider the problem of finding a maximum number of disjoint bases $S_1, S_2, \ldots, S_m$. Let m^* be the maximum achievable for f. An upper bound on m^* is given by the

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[1] A base in a matroid is also required to be minimal and hence independent but minimality is not required for our purposes.

following:

$$k^* = \max\{k : \sum_{i \in N} f_A(i) \geq k f_A(N)\}, \forall A \subseteq N\}.$$

Indeed, $m^* \leq k^*$, since if S is a base, then for all $A \subseteq N$, $f_A(N) = f(N) - f(A) = f(S) - f(A) = f_A(S) \leq \sum_{i \in S} f_A(i)$. The last inequality follows immediately from the submodularity of f_A. By summing over k disjoint bases, we get $\sum_{i \in N} f_A(i) \geq k f_A(N)$. For matroids f, Edmonds [9] showed that $m^* = k^*$ and gave a polynomial time algorithm to obtain m^* disjoint bases. For polymatroids, the situation is different: there are polymatroids f where $m^* \leq (1 + o(1))k^*/\ln f(N)$ [11]. Here, as in the rest of the paper, the little-oh notation is with respect to $f(N)$. Our main result is the following.

Theorem 1.1 *For every polymatroid $f : 2^N \rightarrow \mathcal{Z}^+$, $m^* \geq (1 - o(1))k^*/\ln f(N)$. Moreover, given an oracle access to f, there is a randomized polynomial time algorithm that finds $(1 - o(1))k^*/\ln f(N)$ bases with high probability. There is a deterministic algorithm that, for every fixed $\epsilon > 0$, finds $(1 - \epsilon)k^*/\ln f(N)$ bases.* [2]

As an example, consider a polymatroid that arises from a set coverage system. We are given $|N|$ sets $X_1, X_2, \ldots, X_{|N|}$, each a subset of a universe $\mathcal{U}$. For $S \subseteq N$, let $f(S) = |\bigcup_{j \in S} X_j|$; this is a monotone submodular function. Then, k^* can be interpreted as the minimum coverage of an element; $k^* = \delta = \min_{a \in \mathcal{U}} |\{j : a \in X_j\}|$. In this case, our theorem states that there are at least $(1 - o(1))\delta/\ln n$ disjoint set covers, where $n = |\mathcal{U}|$ and δ the minimum coverage. This result follows immediately from [11].

We prove Theorem 1.1 using a simple argument based on a random permutation. We derandomize the algorithm using ϵ-approximate minwise independent permutations. One could also obtain a proof using a random coloring algorithm used in [11] for the domatic partition problem and a probabilistic lemma on submodular functions from a recent work of the authors [3]. The random permutation based algorithm is easier to analyze and derandomize. We also briefly sketch how the theorem generalizes to the capacitated case in a straightforward way.

One of the motivations for proving Theorem 1.1 is the result of obtained in [5] on a maximum number of disjoint spanning hypergraphs. The random coloring algorithm in [5] is very similar to that in [11] for the domatic partition problem. Both problems can be recast as finding disjoint bases in a polymatroid and Theorem 1.1 can be used to rederive the packing results in [11, 5] in a unified way.

Since the domatic partition problem is a special case of our problem, from the results in in [11], it follows that unless $NP \subseteq DTIME(n^{\log \log n})$, m^* is hard to approximate to within a factor of $(1 - \epsilon)\ln f(N)$ for any fixed $\epsilon > 0$. In [11], the hardness for domatic partition is shown via involved techniques that were used to obtain similar hardness for the corresponding optimization problem, namely the minimum dominating set problem [10]. We observe in Section 5 that a known connection between fractional packings and optimization via linear programming duality can be used to prove hardness for integer packing problems via hardness of a corresponding optimization problem. This gives a simple proof of the hardness of the domatic partition problem and hence for m^*.

The parameter k^* has a natural interpretation in special cases, nevertheless it turns out in general to be NP-hard to compute. In fact we show that there is a constant $c > 0$ such that it is NP-hard to approximate the

[2] Technically, we should take the floor of all our upper-bounds. E.g., if $k^*/\ln f(N) = 1.9$, we guarantee only one base. The floor is needed, as hypergraph 2-coloring can be cast in our framework, similar to the domatic partition as explained in Subsection 3.1, and Erdős [8] probabilistically constructs examples with $k^* = q$, $f(N) = (1 + o(1))\frac{e \ln 2}{4} q^2 2^q$, and where two disjoint bases do not exist.

parameter k^* to within an $(c \log f(N))$-factor: this is essentially the same hardness as computing m^*! The proof is shown in Section 4.

2 Proof of Theorem 1.1

We assume that the value of k^* is known. We cannot compute k^* efficiently, however our algorithm is guaranteed to obtain $(1-o(1))k/\ln f(N)$ bases for any value of $k \leq k^*$ and hence we can try all the values for k in the range $1, 2, \ldots, |N|$ to obtain the desired result. We could also do a binary search which is useful in the capacitated case.

Let $k = \lfloor k^*/(\ln f(N) + \ln\ln f(N)) \rfloor$. If $k \leq 1$ we simply output N as a single base. Otherwise, pick a random permutation of the elements of N, and partition them into at least k parts by putting together exactly $\lfloor |N|/k \rfloor$ consecutive elements.

Fix one such part of $\lfloor |N|/k \rfloor$ elements, let e_i be its i^{th} element in the given permutation, and let $A_i = \{e_1, \ldots,\ e_i\}$, with $A_0 = \emptyset$. A_i and e_i are random choices. It is immediate that for $i = 1, \ldots, \lfloor |N|/k \rfloor$,

$$f_{A_i}(N) = f_{A_{i-1}}(N) - f_{A_{i-1}}(e_i)$$

and therefore

$$E[f_{A_i}(N)] = E[f_{A_{i-1}}(N)] - E[f_{A_{i-1}}(e_i)]. \tag{1}$$

¿From the definition of k^*, and noting that $f_{A_{i-1}}(e) = 0$ for $e \in A_{i-1}$, we have

$$\sum_{e \in N} f_{A_{i-1}}(e) = \sum_{e \in N \setminus A_{i-1}} f_{A_{i-1}}(e) \geq k^* f_{A_{i-1}}(N),$$

which together with the fact that the choice of e_i is uniform from $N \setminus A_{i-1}$ implies:

$$E[f_{A_{i-1}}(e_i)] \geq \frac{1}{|N|+1-i} k^* E[f_{A_{i-1}}(N)] \geq \frac{k^*}{|N|} E[f_{A_{i-1}}(N)]. \tag{2}$$

¿From this and (1) we deduce:

$$E[f_{A_i}(N)] \leq E[f_{A_{i-1}}(N)](1 - \frac{k^*}{|N|}) \leq E[f_{A_{i-1}}(N)] e^{-k^*/|N|}.$$

Applying induction and the fact that $k^*/|N| \cdot \lfloor |N|/k \rfloor \geq \ln f(N) + \ln\ln f(N) - 1$ (here we used $k^* \leq |N|$), we get:

$$E[f_{A_{\lfloor |N|/k \rfloor}}(N)] \leq e^{-\frac{k^*}{|N|} \cdot \lfloor |N|/k \rfloor} f(N) \leq \frac{e}{\ln f(N)}.$$

Since f is integer-valued, Markov's inequality gives:

$$Pr[f_{A_{\lfloor |N|/k \rfloor}}(N) > 0] \leq \frac{e}{\ln f(N)}$$

This holds for a fixed part $A_{\lfloor |N|/k \rfloor}$, and this part is a base when $f_{A_{\lfloor |N|/k \rfloor}}(N) = 0$. Let q be the random variables giving the number of parts which are bases. By linearity of expectations, we obtain that

$$E[q] \geq \left(1 - \frac{e}{\ln f(N)}\right) k \geq \frac{(1-o(1))k^*}{\ln f(N)}.$$

To obtain the result with high probability, one has to notice that q cannot exceed k^*, and repeat the experiment a polynomial number of times.

A Deterministic Algorithm: We obtain a deterministic algorithm by derandomizing the random permutation based algorithm using approximately minwise independent family of permutations. A *minwise* independent family of permutations on $\{1, 2, \ldots, n\}$ is a subset $\mathcal{F} \subseteq S_n$ (S_n is the family of all permutations on $\{1, 2, \ldots, n\}$) with the following property: for a permutation picked at random from $\mathcal{F}$, for every $X \subseteq \{1, 2, \ldots, n\}$ and $x \in X$, the probability that x is the first element in X in the chosen permutation is exactly equal to $1/|X|$. These permutations were defined in [1] for their applications in obtaining sketches for checking similarity in documents. In [1], exponential lower bounds are shown on the size of any minwise independent family. For our purposes, it suffices to have a family that is ϵ-approximate minwise independent; $\mathcal{F}$ is such a family if the probability that $x \in X$ is the first element from X in a randomly chosen permutation from $\mathcal{F}$ is at most $(1+\epsilon)/|X|$ and at least $(1-\epsilon)/|X|$ for every subset X and $x \in X$. For each fixed $\epsilon > 0$, there exists a polynomial sized family of ϵ-approximate minwise indepenent permutations as shown in [14] (see also [16]). Moreover, given ϵ, the family can be explicitly constructed in polynomial time.

Let $\mathcal{F}$ be an ϵ-approximate minwise independent family of permutations. Suppose we chose a random permutation from $\mathcal{F}$ in the algorithm instead of choosing a random permutation from S_n the set of all permutations. We rework the analysis highlighting only the main points. We observe that (1) remains valid. However, (2) is no longer valid since e_i is not uniform in $N \setminus A_{i-1}$. Still, conditioned on A_{i-1}, e_i is approximately uniform in $N \setminus A_{i-1}$. More precisely, from the ϵ-approximate minwise independent property of $\mathcal{F}$, the probability that $e \in N \setminus A_{i-1}$ is e_i is in the interval $[\frac{1-\epsilon}{|N|-i+1}, \frac{1+\epsilon}{|N|-i+1}]$. Thus we obtain the following in place of (2).

$$E[f_{A_{i-1}}(e_i)] \geq \frac{1-\epsilon}{|N|+1-i} k^* E[f_{A_{i-1}}(N)] \geq \frac{(1-\epsilon)k^*}{|N|} E[f_{A_{i-1}}(N)]. \tag{3}$$

The rest of the analysis is essentially the same. To ensure that $Pr[f_{A_{\lfloor N/k \rfloor}}(N) > 0] \leq \frac{e}{\ln f(N)}$ we choose $k = \frac{(1-\epsilon)k^*}{\ln f(N) + \ln \ln f(N)}$. Thus, the the number of bases we obtain in expectation is, as before,

$$E[q] \geq \left(1 - \frac{e}{\ln f(N)}\right) k \geq \frac{(1-2\epsilon)k^*}{\ln f(N)}$$

for sufficiently large $f(N)$.

The above proves that there exists a permutation in $\mathcal{F}$ that yields $(1-2\epsilon)k^*/\ln f(N)$ bases. Since $\mathcal{F}$ is of polynomial size (in $|N|$) and is explicilty available, we try all permutations in $\mathcal{F}$ and obtain the desired number of bases.

This finishes the proof of Theorem 1.1.

The capacitated problem: We briefly discuss the capacitated problem. We are given non-negative integer capacities $u : N \to \mathcal{Z}^+$ and the goal is to output a maximum number of bases such that no element i is contained in more than u_i bases. Let $m^*(u)$ denote the base packing number for a given capacity vector u. The upper bound k^* generalizes in the natural way to $k^*(u)$ in the capacitated setting as follows:

$$k^*(u) = \max\{k : \sum_{i \in N} u_i f_A(i) \geq k f_A(N)\}, \forall A \subseteq N\}.$$

One can reduce the capacitated problem to the uncapcatitated problem by creating u_i copies of i for each $i \in N$ and hence it follows that that $m^*(u) \geq (1 - o(1))k^*(u)/\ln f(N)$. The algorithmic problem of obtaining the bases requires some more care since the reduction to the uncapacitated case is not polynomial.

However, one can use standard and simple scaling ideas that we sketch below. The output is specified in a compact form as a set of bases with corresponding integer multiplicities.

Let $u_{\min} = \min_i u_i$ and $u_{\max} = \max_i u_i$ and let $n = |N|$. First, we observe that $u_{\min} \leq k^* \leq n u_{\max}$. As in the uncapacitated case, the algorithm guesses k^* using binary search (in the interval $[u_{\min}, n u_{\max}]$). The rest of the discussion is for a given guess k^*. If $k^* \leq n^3$ then we can use the reduction to the uncapacitated setting. Otherwise let k be the largest integer in $[0..k^*]$ such that k is divisible by n^2. Note that $k > k^* - n^2$ and by our assumption that $k^* > n^3$, we have $k \geq (1 - 1/n)k^*$. We work with k in place of k^*. This will result in the loss of at most a $(1 - 1/n)$ factor in the number of bases. We assume without loss of generality that $u_i \leq k$ for all i; if $u_i > k$, we can set $u_i = k$. We obtain a a modified capacity u' by setting

$$u'_i = \lfloor \frac{n^2 u_i}{k} \rfloor \quad i \in N.$$

The modified capacities satisfy the property that $0 \leq u'_i \leq n^2$ for all i. We can now apply the reduction to the uncapacitated case. Let m be the number of bases obtained in the modified instance. We obtain mk/n^2 bases in the original instance by creating k/n^2 copies (note that k/n^2 is an integer by the choice of k) of each base in the modified instance. It can be easily checked that scaling loses only a multiplicative factor of $(1 - 2/n)$ in the number of bases.

3 Applications

We show that two results in the literature [11, 5] can be cast as special cases of Theorem 1.1.

3.1 Domatic Number

Given a universe $\mathcal{U}$ and m subsets $X_1, X_2, \ldots, X_m$ of $\mathcal{U}$, define f on the set $N = \{1, 2, \ldots, m\}$ by $f(S) = |\cup_{i \in S} X_i|$. It is easy to check that f is a polymatroid. A special case is when $\mathcal{U}$ is the set of vertices of an undirected graph $G = (V, E)$, and for $v \in V$, we have $X_v = \{v\} \cup \{u \in V \mid (u, v) \in E\}$; thus both $\mathcal{U}$ and N are equal to V, and $f(N) = |V| = n$. Then a polymatroid base is a dominating set in G and the problem of partitioning V into bases is the well-studied DOMATIC NUMBER problem. For this polymatroid k^* can be computed and is equal to $\delta + 1$, where δ is the minimum degree of the graph. Feige et al. [11] use a random coloring to obtain $(1 - o(1))\delta / \ln n$ disjoint dominating sets, and show how their method can be derandomized using conditional probabilities.

For $f(S) = |\cup_{i \in S} X_i|$ derived from an arbitrary set system, we obtain $k^* = \delta$ where $\delta = \min_{a \in U} |\{i \in N : a \in X_i\}|$ is the minimum coverage of an element. Let $n = |\mathcal{U}|$ denote the size of the universe. It can be proved by a random coloring along the lines of [11] that there exist $(1 - o(1))\delta / \ln n$ disjoint collections of sets X_i, each of which covers the entire universe. This is also a special case of our main theorem.

3.2 Disjoint Connected Hypergraphs

Let $H = (V, E)$ be a hypergraph; that is its edges $e \in E$ are arbitrary subsets of V. Given a set of edges $Q \subseteq E$, let $c(Q)$ be the number of connected components of (V, Q), and define $f(Q) = |V| - c(Q)$. It is easy to check that f is a polymatroid on E, and a base of f corresponds to a connected subgraph of H. Partitioning the above polymatroid into bases was considered implictly in [5], where a $\Theta(1/\ln(|V|))$-approximation is given. We improve the constant in the Θ and provide, in our opinion, a cleaner and simpler proof. We remark that the main result in [5] is to pack element disjoint Steiner trees in a graph. They reduce the problem, using a graph reduction step, to the case when G is a bipartite graph with terminals on one side of the bipartition and non-terminals (Steiner nodes) on the other side. Although it is not explicitly stated in [5], finding element disjoint Steiner trees in the above bipartite graph is the same as finding edge-disjoint connected hypergraphs; the hypergraph induced by the bipartite graph has a node for each terminal and a (hyper)edge for each Steiner node.

Assume for simplicity that H is connected. Let q be the size of the minimum global cut in H. Cheriyan and Salavatipour [5] use q in their proof. We do not know how to compute k^*, but we show below that $\frac{1}{2}q \le k^* \le q$.

To show $k^* \le q$, consider a set $Q \subseteq E$ with $|Q| = q$, and $c(E \setminus Q) > 1$; Q is a minimum global cut in H. Let $A = E \setminus Q$; then $f_A(E) = c(A) - 1$ and for all $e \in Q$, we have $f_A(e) \le c(A) - 1$. Then $\sum_{e \in E} f_A(e) = \sum_{e \in Q} f_A(e) \le q(c(A) - 1) = q f_A(E)$; this shows that $k^* \le q$.

To show $k^* \ge \frac{1}{2}q$, consider an arbitrary $A \subseteq E$. Let $V_1, V_2, \ldots, V_j$ be the connected components of the hypergraph induced by A. For any edge $e \in E$, let j_e be the number of these connected components that e intersects. For $i = 1, 2, \ldots, j$, let Q_i be the set of edges e with $e \not\subseteq V_i$ and $e \cap V_i \neq \emptyset$. Each Q_i is a cut and therefore $|Q_i| \ge q$. We conclude with:

$$\sum_{e \in E} f_A(e) = \sum_{e \in E \setminus A} f_A(e) = \sum_{e \in E \setminus A} (j_e - 1) \ge \sum_{e \in E \setminus A} \frac{1}{2} j_e = \frac{1}{2} \sum_{i=1}^{j} |Q_i| \ge \frac{1}{2} q(j-1) = \frac{1}{2} q f_A(E).$$

4 Hardness of computing k^*

Theorem 4.1 *There is $c > 0$ and a family of polymatroids $\mathcal{F}$ such that given $f \in \mathcal{F}$, it is NP-hard to distinguish between $k^* = 1$ and $k^* \ge c \log f(N)$. Moreover, for each $f \in \mathcal{F}$, f can be explicitly evalued in polynomial time.*

We show NP-hardness by reduction to the Vertex Cover problem in hypergraphs. It is known that it is NP-hard to approximate the minimum vertex cover in a r-uniform-hypergraph to within a factor of $(r - 1 - \epsilon)$ [7]. More specifically, it is NP-hard to distinguish between the case where there is a vertex cover of size $n/(r - 1 - \epsilon)$, and the case where any vertex cover has size at least $(1 - \epsilon)n$. It is enough for our purposes to use the following weaker version:

For a given r-uniform hypergraph H on n vertices, $r \ge 4$, it is NP-hard to distinguish between the case where there is a vertex cover smaller than $\frac{3}{5}n$ and the case where any vertex cover has size at least $\frac{4}{5}n$.

This hardness result holds even for superconstant r, up to $r = c \log m$, where m is the number of edges in H and $c > 0$ some constant. This follows relatively easily from known results [13, 6, 7]; we show the details in the appendix. Given a r-uniform hypergraph H, $r = c \log m$, we define a polymatroid f such that $k^* = 1$ in case H has a vertex cover smaller than $\frac{3}{5}n$, and $k^* \ge r/2$ in case every vertex cover has size at least $\frac{4}{5}n$. This will prove that distinguishing between $k^* = 1$ and $k^* \ge r/2$ is NP-hard. In addition, f will satisfy the property that $f(N) = (1 + o(1))m$, i.e. $r = c \log m = (1 - o(1)) c \log f(N)$ which are the desired properties for Theorem 4.1.

Let $H = (V, E)$ the given r-uniform-hypergraph with n denoting $|V|$. The polymatroid $f : 2^V \to \mathcal{Z}^+$ is defined as follows

$$f(S) = g(|S|) + h(S)$$

where $g(s) = \min\{s, \frac{4}{5}n\}$ and $h(S)$ denotes the number of edges in E covered (i.e. hit) by S in H. (We assume for simplicity that n is divisible by 5. This is actually the case for the hard instances of H that we present in Appendix A.) It is easy to see that g and h are submodular and hence f is submodular. The role of $g(s)$ is to make the value of k^* reflect the threshold of $\frac{4}{5}n$ for vertex cover. Recall the definition of $k^* = \max\{k : \forall S; \sum_{i \in N} f_S(i) \geq k f_S(N)\}$. Let us define

$$f_k(S) = f(S) + \frac{1}{k} \sum_{i \in N} f_S(i).$$

I.e., $k^* = \max\{k : \forall S; f_k(S) \geq f(N)\}$. Note that $f_1(S) \geq f(N)$ for any $S \subseteq N$, therefore k^* is at least 1. In general, we have

$$f_k(S) = g(s) + h(S) + \frac{1}{k}(n-s)(g(s+1) - g(s)) + \frac{1}{k} \sum_{i \in N} h_S(i)$$

where $s = |S|$. Note that each edge uncovered by S contributes to r marginal values $h_S(i)$, since there are r vertices potentially covering it. Thus

$$f_k(S) \quad = \quad g(s) + h(S) + \frac{1}{k}(n-s)(g(s+1) - g(s)) + \frac{r}{k}(|E| - h(S)) = g_k(s) + \frac{r}{k}|E| - \frac{r-k}{k} h(S)$$

where we set $g_k(s) = g(s) + \frac{1}{k}(n-s)(g(s+1) - g(s))$. This function is equal to $g_k(s) = s + \frac{1}{k}(n-s)$ for $s < \frac{4}{5}n$, and it is a constant $g_k(s) = \frac{4}{5}n$ for $s \geq \frac{4}{5}n$.

Suppose that $k \leq r/2$, i.e. $\frac{r-k}{k} \geq 1$. Then, if there is an edge uncovered by S, we can always include a new vertex in S, covering a new edge, which increases $g_k(s)$ by at most $1 - 1/k$ and increases $h(S)$ by at least 1. Therefore, $f_k(S)$ decreases; since we want to consider the minimum of $f_k(S)$, we can assume that S is a vertex cover. For a vertex cover, $h(S) = |E|$ and

$$f_k(S) = g_k(s) + |E|.$$

If there is a vertex cover S of size $s < \frac{3}{5}n$, we have $f_2(S) = s + \frac{1}{2}(n-s) + |E| < \frac{4}{5}n + |E| = f(N)$. In this case, $k^* < 2$. Otherwise, we can assume that any vertex cover has size at least $\frac{4}{5}n$. Then we can take $k = r/2$ and we know that the minimum of $f_k(S)$ is attained for a vertex cover, so we have $\min_S f_k(S) \geq \frac{4}{5}n + |E| = f(N)$. So in this case, $k^* \geq r/2$.

5 Hardness of Fractional and Integer Packings

We observed that unless $NP \subseteq DTIME(n^{\log\log n})$, for any $\epsilon > 0$, there is no polynomial time algorithm to obtain a $(1+\epsilon)/\ln f(N)$-approximation to the number of disjoint bases for f. This follows from the inapproximability of the Domatic Number [11]. The proof of Feige *et al.* involves new techniques combining multi-prover protocols and zero-knowledge proofs. Here, we show that similar results can be obtained directly from the inapproximability of the corresponding minimum-weight base problem, namely the minimum weight dominating set problem, using strong duality and a connection between fractional and integer packings. These observations are reasonably well known in combinatorial optimization. However, the implications are occasionally missed and our goal is explicitly work out the result for domatic partition.

Let us define the more general problem of finding *integer* packings of combinatorial structures [3]. Let N be a ground set of n elements and $\mathcal{S} \subseteq 2^N$ be set of *feasible* solutions. One obtains different problems by defining $\mathcal{S}$ appropriately. We will assume the existence of a polynomial time oracle for the following problem: given a set $S \subseteq N$, decide if $S \in \mathcal{S}$ or not. Further, we also assume that there is a polynomial time algorithm to decide if $\mathcal{S} = \emptyset$ or not. The maximum integer packing problem corresponds to finding a maximum number of disjoint subsets $S_1, S_2, \ldots, S_m$ of N such that $S_i \in \mathcal{S}$ for $1 \leq i \leq m$. More generally, one obtains a capacitated version of the problem where each $e \in N$ has an integer capacity u_e and the goal is to find a maximum number of feasible solutions from $\mathcal{S}$ (we allow a solution to be included with multiplicity) such that each e is in at most u_e sets. One can write this as a large integer program as follows. For each $I \in \mathcal{S}$ we have a variable x_I. Then the IP is:

$$
\begin{aligned}
\max \sum_{I \in \mathcal{S}} x_I & & & \\
\sum_{I : e \in I} x_I & \leq & u_e & \qquad \forall e \in N \\
x_I & \in & \mathcal{Z}^+ & \qquad \forall I \in \mathcal{S}.
\end{aligned}
$$

One obtains a *fractional* packing by letting x_I take on values in $\mathcal{R}^+$, which corresponds to taking the LP relaxation of the above integer program. It is known that the complexity of finding a maximum *fractional* packing of feasible solutions in $\mathcal{S}$ is related, via LP duality, to the complexity of finding a minimum cost structure in $\mathcal{S}$. This can be seen by writing the dual of the LP relaxation above.

$$
\begin{aligned}
\min \sum_{e} u_e y_e & & & \\
\sum_{e \in I} y_e & \geq & 1 & \qquad \forall I \in \mathcal{S} \\
y_e & \geq & 0 & \qquad \forall e \in N.
\end{aligned}
$$

The separation oracle for the dual is the following: given *costs* y_e on the elements of N, find the minimum cost feasible set in $\mathcal{S}$. By strong duality and the equivalence of separation and optimization [12], an optimum fractional packing can be found in polynomial time for a problem defined by $\mathcal{S}$ iff a minimum cost set in $\mathcal{S}$ can be found in polynomial time. More recently, the above equivalence has been extended to the approximation setting [4, 15]. In particular it is shown in [15] that an α-approximation for the maximum fractional packing problem is possible iff there is an α-approximation for the minimium cost problem. The easier direction (that an α-approximation for the minimization problem implies an α-approximation for the fractional packing) had been implicitly used for specific problems much earlier in the literature. The other direction, although not too difficult, had not been explicitly stated before. Note that the equivalence requires treating the problem in a general setting with arbitrary capacities and costs.

What does the above say about the complexity of maximum integer packings? There are several examples that show that finding maximum integer packings can be significantly harder than finding maximum fractional

[3] We remark that the discussion in this section applies equally well to coverings instead of packings.

packings, even when the maximum fractional packing can be computed in polynomial time. However, it is intuitively clear that finding a maximum integer packing is at least as hard finding a maximum fractional packing. One needs to be careful to some extent in that there could be capacities u for which finding a maximum fractional packing might be hard while finding an integer packing might be easy for a trivial reason. Therefore, in this discussion, we assume that u is part of the input. In this setting we will prove that an α-approximation for the maximum integer packing problem implies an $\alpha(1-\epsilon)$ approximation for finding a maximum fractional packing where ϵ can be chosen to be as small as $1/\text{poly}(n)$. The implication of this is that the approximability of the integer packing problem is no easier than that of the fractional packing problem. Combining this with the equivalence of the approximability of fractional packing with that of minimization, we can conclude the following. If the minimization problem over $\mathcal{S}$ is hard to approximate to within a factor of α then finding a maximum integer packing over $\mathcal{S}$ is hard to approximate within a factor of α.

We now prove the fact stated above: an α-approximation for the maximum integer packing problem implies an $(1-\epsilon)\alpha$ approximation for finding a maximum fractional packing. The idea is simple; if u is scaled by a large enough factor, integer packing is approximately equivalent to fractional packing and hence we can use an integer packing algorithm to obtain a good fractional packing. Let us fix N and $\mathcal{S}$. Given an integer vector u, let $\nu(u)$ and $\gamma(u)$ be the maximum fractional and integer packing numbers for u. It is obvious that $\nu(\lambda u) = \lambda\nu(u)$ for any $\lambda \geq 0$. We will show the following: $\gamma(u) \geq (1-\epsilon)\nu(u)$ if $u_{\min} = \min_e u_e \geq c(\epsilon)\ln n$ for some function c. To approximate $\nu(u)$ to within a $(1-\epsilon)$ factor we approximate $\gamma(\lambda u)$ where $\lambda \geq c(\epsilon)\ln n$ for an appropriately chosen function c. It suffices to prove that $\gamma(\lambda u) \geq (1-\epsilon)\nu(\lambda u)$. This is a straightforward application of the Chernoff-Hoeffding bounds. Let x be an optimum solution for the fractional packing problem with capacities λu. First, by choosing a basic solution the linear program, the support of x has at most n sets. We can assume without loss of generality that $\nu(u) \geq 1$ and hence $\sum_I x_I = \nu(\lambda u) \geq \lambda$. We assume for simplicity that $x_I \in [0,1]$ for each $I \in \mathcal{S}$ (otherwise we can work with the fractional and integer parts separately). We obtain an integer packing $\hat{x}$ by independently setting, for each I, $\hat{x}_I = 1$ with probability $x_I(1-\epsilon/2)$ and $\hat{x} = 0$ otherwise. By standard analysis, since $\lambda u_e \geq \lambda \geq c(\epsilon)\ln n$ for each e and $\sum_i x_I \geq \lambda = c(\epsilon)\ln n$, with high probability $\hat{x}$ is a feasible integer packing such that $\sum_I \hat{x}_I \geq (1-\epsilon)\sum_I x_I = (1-\epsilon)\nu(\lambda u)$. Finally, we can scale this integer packing by $1/\lambda$ to obtain a fractional packing approximating the optimum with capacities u:

$$\nu(u) = \frac{1}{\lambda}\nu(\lambda u) \geq \frac{1}{\lambda}\gamma(\lambda u) \geq \frac{1-\epsilon}{\lambda}\nu(\lambda u) = (1-\epsilon)\nu(u).$$

The function $c(\epsilon)$ can be chosen to be c'/ϵ^2 for a sufficiently large constant c'. Thus, one can choose $\epsilon = 1/\text{poly}(n)$ while keeping λ polynomial in n.

Consequence for Domatic Number: The main purpose of the above discussion is to show that one can conclude hardness of approximation for integer packings from the hardness of approximation for the corresponding optimization problem. We illustrate this for the domatic partition problem considered in [11] which is the problem of finding a maximum packing of dominating sets in a n-vertex graph. Consider the capacitated version of the domatic partition problem where each vertex has an integer capacity k_v and we seek to find a maximum number of dominating sets such that no vertex is in more than k_v dominating sets. From the above discussion it follows that the capacitated domatic partition problem is hard to approximate to within a factor of $(1-\epsilon)\ln n$ unless $NP \subseteq DTIME(n^{\ln\ln n})$ simply by invoking the hardness of approximating the dominating set problem [10]. We observe that the capacitated domatic partition problem can be reduced to the uncapacitated domatic partition problem by replacing each vertex of capacity k_v by a clique of size k_v and connecting two vertices in different cliques iff the original vertices corresponding to the cliques were adjacent. This reduction is not polynomial; however one can use simple scaling tricks that allows one to assume that the maximum capacity is polynomial in the number of vertices (with a loss of a $(1-\epsilon)$ factor in the approximation). Therefore the domatic partition problem is hard to approximate to $(1-\epsilon)\ln n$ unless $NP \subseteq DTIME(n^{\ln\ln n})$.

6 Conclusions

In [11] an $O(\ln \Delta(G))$ approximation is given for the domatic partition problem where $\Delta(G)$ is the maximum degree in the given graph G. This ratio improves the $(1-o(1))\ln n$ approximation when $\Delta(G)$ is small compared to n. The corresponding question in the polymatroid setting is the following: is there an $O(\ln \Delta(f))$ approximation for m^* where $\Delta(f) = \max_{i \in N} f(i)$? Wolsey [17] showed that the greedy algorithm gives a $(1 + \ln \Delta(f))$ approximation for the minimum weight base in a polymatroid f. From the duality discussed in Section 5, a $(1 + \ln \Delta(f))$ approximation for the maximum fractional packing follows. Can we obtain a similar bound for integer packings? The bound in [11] is obtained using the Lovász Local Lemma and takes advantage of the graph structure. For general polymatroids a similar randomized algorithm does not work. Nevertheless we do not know of any f for which $m^* = o(k^*/\ln \Delta(f))$.

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A A hardness result for vertex cover in hypergraphs

In Section 4, we used the following hardness result:

For a given r-uniform hypergraph H with n vertices and m edges, $r = c \log m$ for some constant $c > 0$, it is NP-hard to distinguish between the case where there is a vertex cover smaller than $\frac{3}{5}n$ and the case where any vertex cover has size at least $\frac{4}{5}n$.

Here we show how this follows from results stated explicitly in the literature [13, 6, 7]. (We are indebted to Julia Chuzhoy for pointing out this reduction to us.) We borrow the following result from [6]:

Theorem A.1 ([6]) *For any $\delta > 0$, it is NP-hard to approximate Vertex Cover in 4-uniform hypergraphs to within $2-\delta$. In particular, it is NP-hard to distinguish between the minimum vertex cover being at most $(\frac{1}{2}+\delta)n$ and at least $(1-\delta)n$, where n is the number of vertices.*

(The first part of the above theorem corresponds to Theorem 3.1 in [6] and the second part of the statement is stated in two lemmas immediately following Theorem 3.1 in [6].)

For a given 4-uniform hypergraph H and a given parameter $d \geq 1$, we perform the following operation to define a new hypergraph H'. We replace each vertex v of H by a set of $10d$ "clones" C_v. For each 4-edge $\{v_1, v_2, v_3, v_4\} \in E(H)$, we define $\binom{10d}{d}^4$ hyperedges of H', where each hyperedge is formed by choosing d clones for each of the 4 vertices v_1, v_2, v_3, v_4: $e = A_1 \cup A_2 \cup A_3 \cup A_4$, $A_i \subset C_{v_i}$, $|A_i| = d$. If H had n vertices and m edges, H' now has $n' = 10dn$ vertices and $\binom{10d}{d}^4 m = 2^{O(d)} m$ hyperedges. We choose $d = \log m$, so that H' has $m' = poly(m)$ hyperedges of size $r' = 4d = \Omega(\log m')$.

We show that it is NP-hard to distinguish whether the minimum vertex cover in H' is smaller than $\frac{3}{5}n'$ or at least $\frac{4}{5}n'$. Consider $\delta < \frac{1}{10}$ and the two cases which are NP-hard to distinguish due to Theorem 3.1 [6]:

- If there is a vertex cover S in H of size $|S| \leq (\frac{1}{2} + \delta) < \frac{3}{5}n$, then we can take all the clones for each vertex in S. This gives a vertex cover $S' = \bigcup_{v \in S} C_v$ of size $10d|S| < 6dn = \frac{3}{5}n'$.
- If any vertex cover in H has size at least $(1-\delta)n > \frac{9}{10}n$, consider any set S' of $\frac{4}{5}n' = 8dn$ vertices in H'. Let $W \subseteq V(H)$ denote vertices that have at least $9d$ clones in S'. Since $9d|W| \leq |S'| = 8dn$, we have $|W| \leq \frac{8}{9}n$. Therefore, W cannot be a vertex cover in H. Consider an edge $\{v_1, v_2, v_3, v_4\}$ disjoint from W. Since each v_i has at most $9d$ clones in S', there is still a set of d clones $A_i \subset C_{v_i}$ disjoint from S'. We have found a hyperedge $A_1 \cup A_2 \cup A_3 \cup A_4 \in E(H')$ disjoint from S', i.e. S' is not a vertex cover in H'. We have proved that there is no vertex cover of size $\frac{4}{5}n'$.

Remark. Note that this construction does not produce r' larger than $O(\log m')$, even for larger d, because m' grows exponentially with d. Indeed, the hardness result stated here (with a gap between $\frac{3}{5}n$ and $\frac{4}{5}n$) does not hold for $\omega(\log m)$-uniform hypergraphs. This is not surprising, since for an r-uniform hypergraph with m hyperedges, where $r = c \log m$, $c > 2$, taking each vertex independently with probability $2/c$ gives a vertex cover with high probability.