Distributed Construction of a Planar Spanner and Routing for Ad Hoc Wireless Networks

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Abstract—**Several localized routing protocols [1] guarantee the delivery of the packets when the underlying network topology is the Delaunay triangulation of all wireless nodes. However, it is expensive to construct the Delaunay triangulation in a distributed manner. Given a set of wireless nodes, we more accurately model the network as a unit-disk graph UDG, in which a link in between two nodes exist only if the distance in between them is at most the maximum transmission range.**

Given a graph H, a spanning subgraph G of H is a t-spanner if the length of the shortest path connecting any two points in G is no more than t times the length of the shortest path connecting the two points in H. In this paper, we present a novel localized networking protocol that constructs a planar 2.5-spanner of UDG, called the localized Delaunay triangulation, as network topology. It contains all edges that are both in the unit-disk graph and the Delaunay triangulation of all wireless nodes.

Our experiments show that the delivery rates of existing localized routing protocols are increased when localized Delaunay triangulation is used instead of several previously proposed topologies. The total communication cost of our networking protocol is $O(n \log n)$ bits. Moreover, the computation cost of each node u is $O(d_u \log d_u)$, where d_u is the number of 1-hop neighbors of u in UDG.

I. INTRODUCTION

In a wireless ad hoc network (or sensor network), assume that all wireless nodes have distinctive identities and each static wireless node knows its position information, either through a low-power Global Position System (GPS) receiver or through some other way. For simplicity, we also assume that all wireless nodes have the same maximum transmission range and we normalize it to one unit. By a simple broadcasting, each node u can gather the location information of all nodes within the transmission range of u. Consequently, all wireless nodes S together define a unit-disk graph $UDG(S)$, which has an edge uv if and only if the Euclidean distance $\|uv\|$ between u and v is less than one unit.

One of the central challenges in the design of *ad hoc* networks is the development of dynamic routing protocols that can efficiently find routes between two communication nodes. In recent years, a variety of routing protocols [2], [3], [4], [5], [6], [7], [8] targeted specifically for *ad hoc* environment have been developed. For the review of the state of the art routing protocols, see surveys by E. Royer and C. Toh [9] and by S. Ramanathan and M. Steenstrup [10].

Several researchers proposed another set of routing protocols, namely the localized routing, which select the next node to forward the packets based on the information in the packet header, and the position of its local neighbors. Bose and Morin [1] showed that several localized routing protocols guarantee to deliver the packets if the underlying network topology is the Delaunay triangulation of all wireless nodes. They also gave a localized routing protocol based on the Delaunay triangulation such that the total distance traveled by the packet is no more than a small constant factor of the distance between the source and the destination. However, it is expensive to construct the Delaunay triangulation in a distributed manner, and routing based on it might not be possible since the Delaunay triangulation can contain links longer than one unit. Then, several researchers proposed to use some *planar* network topologies that can be constructed efficiently in a distributed manner. Lin *et al.* [11], Bose *at al.*[12] and Karp *et al.* [13] proposed to use the Gabriel graph. Routing according to the right hand rule, which guarantees delivery in planar graphs [1], is used when simple greedy-based routing heuristics fail.

Given a graph H, a spanning subgraph G of H is a t-spanner if the length of the shortest path connecting any two points in G is no more than t times the length of the shortest path connecting the two points in H. In this paper, we design a localized algorithm that constructs a planar t-spanner for the unit-disk graph $UDG(S)$, such that some of the localized routing protocols can be applied on it. We obtain a value of approximately 2.5 for the constant t.

Given a set of points S, let $UDel(S)$, the *unit Delaunay triangulation*, be the graph obtained by removing all edges of $Del(S)$ that are longer than one unit. We first prove that $UDel(S)$ is a t-spanner of the unit-disk graph $UDG(S)$. We then give a localized algorithm that constructs a graph, called localized Delaunay graph $LDel^{(1)}(S)$. We prove that $LDel^{(1)}(S)$ is a t-spanner by showing that it is also a supergraph of $UDel(S)$. We then show how to make the graph $LDel^{(1)}(S)$ planar efficiently. The total communication cost of our approach is $O(n \log n)$ bits, which is optimal within a constant factor.

Bose *et al.* [12] and Karp *et al.* [13] proposed similar algorithms that route the packets using the Gabriel graph to guarantee the delivery. Applying the routing methods proposed in [12], [13] on the planarized localized Delaunay graph $LDel^{(1)}(S)$, a better performance is expected because the localized Delaunay triangulation is denser compared to the Gabriel graph, but still with $O(n)$ edges. Our simulations show that the delivery rates of several localized routing protocols are increased when the localized Delaunay triangulation is used. In our experiments, several simple local routing heuristics, applied on the localized Delaunay triangulation, have always successfully delivered the packets, while other heuristics were successful in over 90% of the random instances. Moreover, because the constructed topology is planar, a localized routing algorithm using the right hand rule guarantees the delivery of the packets from source node to the destination when simple heuristics fail. The experiments also show that several localized routing algorithms (notably, com-

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pass routing [14] and greedy routing) also result in a path whose length is within a small constant factor of the shortest path; we already know such a path exists since the localized Delaunay triangulation is a t-spanner.

The remaining of the paper is organized as follows. In Section II, we review some structures that are often used to construct the topology for wireless networks. In Section III, we show that the unit Delaunay triangulation $UDel$ is a t-spanner, where $t = \frac{1+\sqrt{5}}{2}\pi$. We also claim that t can be reduced to $\frac{4\sqrt{3}}{9}\pi \approx 2.42$. We define localized Delaunay triangulations $LDel^{(k)}(S)$ and study their properties in Section IV. Section V presents the first localized efficient algorithm that constructs a planar graph, $PLDel(S)$, which contains $UDel(S)$ as a subgraph. Thus, $PLDel(S)$ is a planar t-spanner. The correctness of our algorithm is justified in the Appendix. We demonstrate the effectiveness of the localized Delaunay triangulation in Section VI by studying the performance of various routing protocols on it. We conclude our paper and discuss possible future research directions in Section VII.

II. PRELIMINARIES

A. Voronoi Diagram and Delaunay Triangulation

We begin with definitions of the Voronoi diagram and the Delaunay triangulation [15]. We assume that all wireless nodes are given as a set S of n nodes in a two dimensional space. Each node has some computational power. We also assume that there are no four nodes of S that are co-circular. A triangulation of S is a *Delaunay triangulation*, denoted by $Del(S)$, if the circumcircle of each of its triangles does not contain any other nodes of S in its interior. A triangle is called the *Delaunay triangle* if its circumcircle is empty of nodes of S. The *Voronoi region*, denoted by $Vor(p)$, of a node p in S is the collection of two dimensional points such that every point is closer to p than to any other node of S. The *Voronoi diagram* for S is the union of all Voronoi regions $Vor(p)$, where $p \in S$. The Delaunay triangulation $Del(S)$ is also the dual of the Voronoi diagram: two nodes p and q are connected in $Del(S)$ if and only if $Vor(p)$ and $Vor(q)$ share a common boundary. The shared boundary of two Voronoi regions $Vor(p)$ and $Vor(q)$ is on the perpendicular bisector line of segment pq. The boundary segment of a Voronoi region is called the *Voronoi edge*. The intersection point of two Voronoi edge is called the *Voronoi vertex*. Each Voronoi vertex is the circumcenter of some Delaunay triangle.

B. Spanner

Constructing a spanner of a graph has been well studied. Let $\Pi_G(u,v)$ be the shortest path connecting u and v in a weighted graph G, and $||\Pi_G(u,v)||$ be the length of $\Pi_G(u,v)$.

Then a graph G is a t-spanner of a graph H if $V(G) = V(H)$ and, for any two nodes u and v of $V(H)$, $||\Pi_H(u,v)|| \leq ||\Pi_G(u,v)|| \leq t||\Pi_H(u,v)||$. With H understood, we also call t the *length stretch factor* of the spanner G. There are several geometrical structures which are proved to be t-spanners for the Euclidean complete graph $K(S)$ of a point set S. For example, the Yao graph [16] and the θ-graph [17] have been shown to be t-spanners. However, both these two geometrical structures are not guaranteed to be planar in two dimensions. Given a set of points S, it is well-known that the Delaunay triangulation $Del(S)$ is a planar t-spanner of the completed Euclidean graph $K(S)$. This is first proved by Dobkin, Friedman and Supowit [18] with upper bound $\frac{1+\sqrt{5}}{2}\pi \approx 5.08$ on t. Then Kevin and Gutwin [19], [17] improved the upper bound on t to be $\frac{2\pi}{3\cos\frac{\pi}{6}} = \frac{4\sqrt{3}}{9}\pi \approx 2.42$. The best known lower bound on t is $\pi/2$, which is due to Chew [20].

C. Proximity Graphs

Let S be a set of n wireless nodes distributed in a two-dimensional plane. These nodes induce a *unit-disk graph* $UDG(S)$ in which there is an edge uv if and only if $||uv|| \leq 1$. Various proximity subgraphs of the unit-disk graph can be defined [21], [22], [23], [24], [16].

For convenience, let $disk(u,v)$ be the closed disk with diameter uv, let $disk(u,v,w)$ be the circumcircle defined by the triangle $\triangle uvw$, and let $B(u,r)$ be the circle centered at u with radius r. Let $x(v)$ and $y(v)$ be the value of the x-coordinate and y-coordinate of a node v respectively.

- The *constrained relative neighborhood graph*, denoted by $RNG(S)$, consists of all edges uv such that $||uv|| \leq 1$ and there is no point $w \in S$ such that $||uw|| < ||uv||$, and $||wv|| < ||uv||$.
- The *constrained Gabriel graph*, denoted by $GG(S)$, consists of all edges uv such that $||uv|| \leq 1$ and $disk(u,v)$ does not contain any node from S.
- The *constrained Yao graph* with an integer parameter $k \geq 6$, denoted by $\overrightarrow{YG}_k(S)$, is defined as follows. At each node u, any k equal-separated rays originated at u define k cones. In each cone, choose the closest node v to u with distance at most one, if there is any, and add a directed link $\overrightarrow{uv}$. Ties are broken arbitrarily. Let $YG_k(S)$ be the undirected graph obtained by ignoring the direction of each link in $\overrightarrow{YG}_k(S)$.

Bose *et al.* [25] showed that the length stretch factor of $RNG(V)$ is at most $n-1$ and the length stretch factor of $GG(V)$ is at most $\frac{4\pi\sqrt{2n-4}}{3}$. Several papers [26], [27], [21] showed that the Yao graph $YG_k(V)$ has length stretch factor at most $\frac{1}{1-2\sin\frac{\pi}{k}}$. However, the Yao graph is not guaranteed to be planar. The relative neighborhood graph and the Gabriel graph are planar graphs, but they are not a spanner for the unit-disk graph. In this paper, we are interested in locally constructing a planar graph that is a spanner of the unit-disk graph. In our experiments, routing packets using several simple localized routing algorithms such as compass routing on this localized Delaunay triangulation was always or almost always successful, improving on routing on the Gabriel graph or the relative neighborhood graph.

D. Localized Routing Algorithms

Let $N_k(u)$ be the set of nodes of S that are within k hops distance of u in the unit-disk graph $UDG(S)$. A node $v \in N_k(u)$ is called the k-neighbor of the node u. Usually, here the constant k is 1 or 2, which will be omitted if it is clear from the context. In this paper, we always assume that each node u of S knows its location and identity. Then, after one broadcast by every node, each node u of S knows the location and identity information of all nodes in $N_1(u)$. The total communication cost of all nodes to do so is $O(n \log n)$ bits.

A distributed algorithm is a *localized algorithm* if it uses only the information of all k-local nodes of each node plus the information of a constant number of additional nodes. In this paper, we concentrate on the case $k = 1$. That is, a node uses only the information of the 1-hop neighbors. A graph G can be constructed locally in the ad hoc wireless environment if each wireless node u can compute the edges of G incident on u by using only the location information of all its k-local nodes. In this paper, we design a localized algorithm that constructs a planar t-spanner for the unit-disk graph $UDG(S)$ such that some localized routing protocols can be applied on it.

Assume a packet is currently at node u, and the destination node is t. Several localized routing algorithms that just use the local information of u to route packets (i.e., find the next node v of u) were developed. Kranakis *et al.* [14] proposed to use the compass routing, which basically finds the next relay node v such that the angle $\angle vut$ is the smallest among all neighbors of u in a given topology. Lin *et al.* [11], Bose *et al.* [12], and Karp *et al.* [13] proposed similar greedy routing methods, in which node u forwards the packet to its neighbor v in a given topology which is closest to t. Recently, Bose *at al.*[28], [1], [12] proposed several localized routing algorithm that route a packet from a source node s to a destination node t. Specifically, Bose and Morin [1] proposed a localized routing method based on the Delaunay triangulation. They showed that the distance traveled by the packet is within a small constant factor of the distance between s and t. They also proved that the compass routing and the greedy routing method guarantee to deliver the packet if the Delaunay triangulation is used.

III. GRAPH $UDel(S)$ IS A SPANNER

In this section, we prove that $UDel(S)$ is a spanner with stretch factor $t = \frac{1+\sqrt{5}}{2}\pi$. We claim the stronger result that $UDel(S)$ is a $\frac{4\sqrt{3}}{9}\pi$-spanner, but omit the proof due to space limitations.

Dobkin, Friedman and Supowit proved that, for any two points u and v of a point set S, the shortest path connecting u and v in the Delaunay triangulation $Del(S)$ has length no more than $\frac{1+\sqrt{5}}{2}\pi\|uv\|$. However, it is not appropriate to require the construction of the Delaunay triangulation in the wireless communication environment because of the possible massive communications it requires. Therefore, we consider the following subset of the Delaunay triangulation. Let $UDel(S)$ be the graph by removing all edges of $Del(S)$ that are longer than one unit, i.e., $UDel(S) = Del(S) \cap UDG(S)$. Call $UDel(S)$ the *unit Delaunay triangulation.* For the remainder of this section, we will prove that $UDel(S)$ is a t-spanner of the unit-disk graph $UDG(S)$.

Our proof is based on the remarkable proof by Dobkin *et al.*[18]. They proved that the Delaunay triangulation is a t-spanner by constructing a path $\Pi_{dfs}(u, v)$ in $Del(S)$ with length no more $\frac{1+\sqrt{5}}{2}\pi\|uv\|$. The constructed path consists of at most two parts: one is some *direct DT* paths, the other is some *shortcut* subpaths.

Given two nodes u and v, let $b_0 = u$, b_1, b_2, $\cdots$, b_{m-1}, $b_m = v$ be the nodes corresponding to the sequence of Voronoi regions traversed by walking from u to v along the segment uv. See Figure 1 for an illustration. If a Voronoi edge or a Voronoi vertex happens to lie on the segment uv, then choose the Voronoi region lying above uv. Assume that the line uv is the x-axis. The sequence of nodes b_i, $0 \le i \le m$, defines a path from u to v. In general, they [18] refer to the path constructed this way between some nodes u and v as the *direct DT path* from u to v. Then Dobkin *et al.* proved the following lemma.

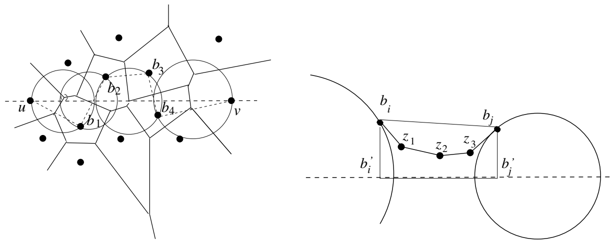

Fig. 1. Left: The direct DT path $ub_1b_2b_3b_4v$ between u and v shown by dashed lines; Right: The short cut from node b_i to node b_j.

Lemma 1: For all i, $0 \le i \le m$, b_i is contained within or on the boundary of $disk(u, v)$.

A stronger result is that all nodes b_i, $0 \le i \le m$, are on the boundary of the union of all circles C_i, $1 \le i \le m$, where $C_i = B(p_i, \|p_i b_i\|)$ and p_i is the point on the x-axis that also lies on the boundary between the Voronoi regions $Vor(b_{i-1})$ and $Vor(b_i)$. The boundary of the union of all circles C_i has length at most $\pi \cdot \|uv\|$; For details, see [18]. This implies that if a direct DT path always lies above (or below) uv, then its length is at most $\frac{\pi}{2} \cdot \|uv\|$. If the direct DT path connecting u and v is lying entirely above or entirely below the segment uv, it is called *one-sided*; see [18].

The Lemma 1 also implies that the distance $\|b_i b_j\|$ between any two nodes b_i and b_j is at most $\|uv\|$. Consequently, we have the following corollary.

Corollary 2: All edges of the direct DT path connecting two nodes s and t have length at most $\|st\|$.

The path constructed by Dobkin *et al.* uses the direct DT path as long as it is above the x-axis. Assume that the path constructed so far has brought us to some node b_i such that $y(b_i) \ge 0$, $b_i \ne v$, and $y(b_{i+1}) < 0$. Let j be the least integer larger than i such that $y(b_j) \ge 0$. Notice that here j exists because $y(b_m) = 0$ by assuming that uv is the x-axis. Then the path constructed by Dobkin *et al.* uses either the direct DT path to b_j or takes a *shortcut* as follows[1]. Construct the lower convex hull $z_0 = b_i, z_1, \cdots, z_{l-1}, z_l = b_j$ of the following set of nodes:

$$\{q \in S \mid x(b_i) \le x(q) \le x(b_j) \text{ and } y(q) \ge 0 \\ \text{and } q \text{ lies under } b_i b_j\ \}.$$

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[1] See [18] for more detail about the condition when to choose the direct DT path from b_i to b_j and when to choose the shortcut path from b_i to b_j.

Notice that except z_0 and z_l, all nodes $z_1, \cdots, z_{l-1}$ do not belong to $\{b_1, b_2, \cdots, b_{m-1}, b_m\}$ and the edges of the convex hull are not on the direct DT path from u to v. The shortcut path consists of taking the direct DT path from z_k to z_{k+1} for each $0 \leq k \leq l-1$, which is shown to be on one side of line $z_k z_{k+1}$ if the shortcut path is chosen.

Dobkin *et al.* then proved that the length of the path traversed from u to v has length at most $\frac{1+\sqrt{5}}{2}\pi\|uv\|$. Similar to the direct DT path, we prove the following lemma.

Lemma 3: All edges of the shortcut path connecting two nodes b_i and b_j have length at most $\|uv\|$.

Proof: Figure 1 gives intuition on the proof that follows. Let b'_i, b'_j be the projection points of nodes b_i and b_j on the x-axis (segment uv), respectively. Then from the definition of z_0, $z_1, \cdots, z_{l-1}, z_l$, we know that z_k, $0 \leq k \leq l$ lies inside or on the boundary of the trapezoid $b_i b_j b'_j b'_i$, which lies inside the $disk(u, v)$. Consequently, edge $z_k z_{k+1}$, for each $0 \leq k \leq l-1$ has length at most $\|uv\|$. From Corollary 2, we know that all edges of the direct DT path from z_k to z_{k+1} have length at most $\|z_k z_{k+1}\|$. Then the lemma follows. ■

Consequently, we have the following lemma.

Lemma 4: Let $\Pi_{dfs}(u, v)$ be the path constructed by Dobkin *et al.* from u to v in the Delaunay triangulation. All edges in $\Pi_{dfs}(u, v)$ have length at most $\|uv\|$.

Then the following theorem is straightforward.

Theorem 5: For any two nodes u and v of S,

$$\|\Pi_{UDel(S)}(u, v)\| \leq \frac{1+\sqrt{5}}{2}\pi \cdot \|\Pi_{UDG(S)}(u, v)\|.$$

Proof: Assume $\Pi_{UDG(S)}(u, v) = v_0 v_1 \cdots v_{h-1} v_h$, where $u = v_0$ and $v = v_h$, is the shortest path connecting u and v in $UDG(S)$. Then for each link $v_i v_{i+1}$, $0 \leq i \leq h-1$, there is a path $\Pi_{Del(S)}(v_i, v_{i+1})$ in the Delaunay triangulation (constructed using the method proposed in [18]) $Del(S)$ with length at most $\frac{1+\sqrt{5}}{2}\pi \cdot \|v_i v_{i+1}\|$. Notice that $\|v_i v_{i+1}\| \leq 1$ and all edges in $\Pi_{Del(S)}(v_i, v_{i+1})$ have length at most $\|v_i v_{i+1}\|$. Therefore each path $\Pi_{Del(S)}(v_i, v_{i+1})$, $0 \leq i \leq h-1$, is also in the graph $UDel(S)$. Then the path formed by concatenating all paths $\Pi_{Del(S)}(v_i, v_{i+1})$, $i = 0, \cdots, h-1$ has length at most $\frac{1+\sqrt{5}}{2}\pi \cdot \|\Pi_{UDG(S)}(u, v)\|$. The theorem follows. ■

Kevin and Gutwin [19], [17] showed that the Delaunay triangulation is a t-spanner for a constant $t = \frac{2\pi}{3\cos\frac{\pi}{6}} = \frac{4\sqrt{3}}{9}\pi \approx 2.42$. They proved this using induction on the order of the lengths of all pair of nodes (from the shortest to the longest). We can show that the path connecting nodes u and v constructed by the method given in [19], [17] also satisfies that all edges of that path is shorter than $\|uv\|$. Due to space limitations, we omit the proof. Consequently, we have:

Theorem 6: $UDel(S)$ is a $\frac{4\sqrt{3}}{9}\pi$-spanner of UDG.

IV. LOCAL DELAUNAY TRIANGULATION

In this section, we will define a new topology, called local Delaunay triangulation, which can be constructed in a localized manner. We first introduce some geometric structures and notations to be used in this section. All angles are measured in radians and take values in the range $[0, \pi]$. For any three points p_1, p_2, and p_3, the angle between the two rays p_1p_2 and p_1p_3 is denoted by $\angle p_3p_1p_2$ or $\angle p_2p_1p_3$. The *closed* infinite area inside the angle $\angle p_3p_1p_2$, also referred to as a sector, is denoted by $\measuredangle p_3p_1p_2$. The triangle determined by p_1, p_2, and p_3 is denoted by $\triangle p_1p_2p_3$.

An edge uv is called *Gabriel edge* if $\|uv\| \leq 1$ and the open disk using uv as diameter does not contain any node from S. It is well known [15] that the constrained Gabriel graph is a subgraph of the Delaunay triangulation. Recall that a triangle $\triangle uvw$ belongs to the Delaunay triangulation $Del(S)$ if its circumcircle $disk(u, v, w)$ does not contain any other node of S in its interior. Here we often assume that there are no four nodes of S co-circumcircle. It is easy to show that nodes u, v and w together can not decide if they can form a triangle $\triangle uvw$ in $Del(S)$ by using only their local information. We say a node x can see another node y if $\|xy\| \leq 1$. The following definition is one of the key ingredients of our localized algorithm.

Definition 1: A triangle $\triangle uvw$ satisfies *k-localized Delaunay property* if the interior of $disk(u, v, w)$ does not contain any node of S that is a k-neighbor of u, v, or w; and all edges of the triangle $\triangle uvw$ have length no more than one unit. Triangle $\triangle uvw$ is called a *k-localized Delaunay triangle*.

Triangle $\triangle uvw$ is called *localized Delaunay* if it is a *k-localized Delaunay triangle* for some integer $k \geq 1$.

Definition 2: The *k-localized Delaunay graph* over a node set S, denoted by $LDel^{(k)}(S)$, has exactly all Gabriel edges and edges of all k-localized Delaunay triangles.

When it is clear from the context, we will omit the integer k in our notation of $LDel^{(k)}(S)$. Our original conjecture was that $LDel^{(1)}(S)$ is a planar graph and thus we can easily construct a planar t-spanner of $UDG(S)$ by using a localized approach. Unfortunately, as we will show later, the edges of the graph $LDel^{(1)}(S)$ may intersect. While $LDel^{(1)}(S)$ is a t-spanner, its construction is a little bit more complicated than some other non-planar t-spanners, such as the Yao structure [16] and the θ-graph [17]. But we can make $LDel^{(1)}(S)$ planar efficiently, a result we describe later in this paper.

Notice that the *k-localized Delaunay graph* $LDel^{(k)}(S)$ over a node set S satisfies a monotone property: $LDel^{(k+1)}(S)$ is always a subgraph of $LDel^{(k)}(S)$ for any positive integer k.

A. $LDel^{(1)}(S)$ *may be non-planar*

The definition of the 1-localized Delaunay triangle does not prevent two triangles from intersecting or prevent a Gabriel edge from intersecting a triangle. Figure 2 gives such an example with 6 nodes $\{u, v, w, x, y, z\}$ that $LDel^{(1)}(S)$ is not a planar graph. Here $\|uv\| = 1$. Triangle $\triangle uvw$ is a 1-localized Delaunay triangle. If the node z does not exist, edge xy is an Gabriel edge. The triangle $\triangle uvw$ intersects the Gabriel edge xy if z does not exist, otherwise it intersects the 1-localized Delaunay triangle $\triangle xyz$.

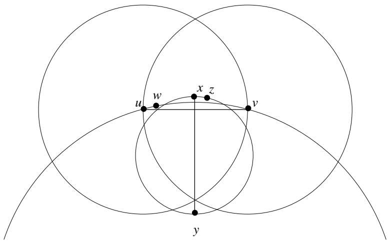


Fig. 2. $LDel^{(1)}(S)$ is not planar.

B. $LDel^{(k)}(S)$ is a t-spanner

Theorem 7: Graph $UDel(S)$ is a subgraph of the k-localized Delaunay graph $LDel^{(k)}(S)$.

Proof: We prove the theorem by showing that each edge uv of the unit Delaunay triangulation graph $UDel(S)$ appears in the localized Delaunay graph $LDel^{(k)}(S)$. For each edge uv of $UDel(S)$, the following five cases are possible (see Figure 3 for illustrations).

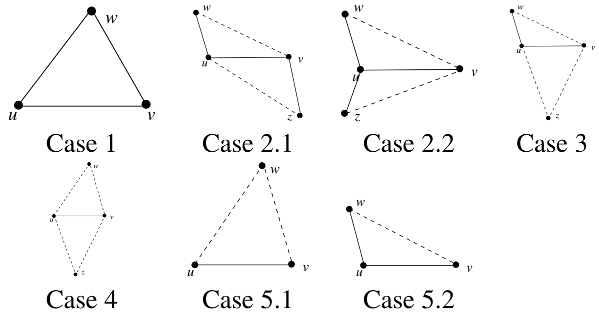


Fig. 3. The neighborhood configuration of edge uv. Dashed lines (solid lines) denote edges with length $> 1 (\leq 1)$.

Case 1: there is a triangle $\triangle uvw$ incident on uv such that all edges of $\triangle uvw$ have length at most one unit. Because the circumcircle $disk(u, v, w)$ is empty of nodes of S, triangle $\triangle uvw$ satisfies the k-localized Delaunay property and thus edge uv belongs to $LDel^{(k)}(S)$.

Case 2: each of the two triangles incident on uv has only one edge with length larger than one unit.

Case 3: one triangle $\triangle uvw$ incident on uv has only one edge with length larger than one unit and the other triangle $\triangle uvz$ has two edges with length larger than one unit.

Case 4: each of the two triangles incident on uv has two edges with length larger than one unit.

We prove the cases 2, 3, and 4 together. Assume the two triangles are $\triangle uvw$ and uvz. Let $H_{uv,w}$ be the half-plane that is divided by uv and contains node w. Then edge uv is not the longest edge in triangle $\triangle uvw$ and thus the angle $\angle uwv < \frac{\pi}{2}$; for an illustration, see Figure 4. This implies that the circumcircle $disk(u, v, w)$ contains $disk(u, v) \cap H_{uv,w}$. Similarly, the other half of $disk(u, v)$ is contained inside the circumcircle $disk(u, v, z)$. Notice that both $disk(u, v, w)$ and $disk(u, v, z)$ do not contain any node of S inside. It implies that $disk(u, v)$ is empty, i.e., edge uv is a Gabriel edge. Consequently, edge uv will be inserted to $LDel^{(k)}(S)$.

Case 5: there is only one triangle incident on uv and it has at least one edge with length larger than one unit. Similar to cases 2-4, we can show that $disk(u, v)$ is empty and therefore edge uv will be inserted to $LDel^{(k)}(S)$ as a Gabriel edge. ■

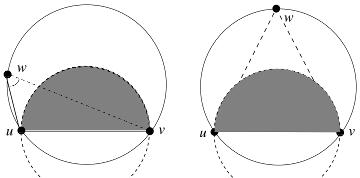


Fig. 4. Gabriel edges.

C. $LDel^{(k)}(S)$, $k \geq 2$, is planar

The above proof implies that each edge uv of $UDel(S)$ is either a Gabriel edge or forms a 1-localized Delaunay triangle with some edges from $UDel(S)$. Any two edges in $UDel(S)$ do not intersect. Thus, each possible intersection in $LDel^{(k)}(S)$ is caused by at least one localized Delaunay triangle. We begin the proof that $LDel^{(k)}(S)$, $k \geq 2$, is planar by giving some simple facts and lemmas.

Lemma 8: If an edge xy intersects a localized Delaunay triangle $\triangle uvw$, then x and y can not be both inside the circumcircle $disk(u, v, w)$.

Proof: For the sake of contradiction, assume that x and y are both inside $disk(u, v, w)$. Notice that $disk(u, v, w)$ is divided into four regions by the triangle $\triangle uvw$. Let $\widehat{uv}$, $\widehat{vw}$, and $\widehat{wu}$ be the three fan regions defined by edges uv, vw, and wu respectively. First of all, neither x nor y can be inside the triangle $\triangle uvw$. Assume that x is inside the region $\widehat{uv}$ and y is inside the region $\widehat{vw}$. Then one of the angles $\angle uwv$ and $\angle vuw$ is less than $\frac{\pi}{2}$, which implies that one of the angle $\angle uxv$ and $\angle vyw$ is larger than $\frac{\pi}{2}$. Thus, either $vy < vw \leq 1$ or $vx < vu \leq 1$. In other words, the $disk(u, v, w)$ contains a node from $N_1(v)$. This contradicts that $\triangle uvw$ is a k-localized Delaunay triangle. ■

Lemma 9: If a Gabriel edge xy intersects a localized Delaunay triangle $\triangle uvw$, then x and y can not be both outside the circumcircle $disk(u, v, w)$.

Proof: Let c be the circumcenter of the triangle $\triangle uvw$. Then at least one of the u, v, and w must be on the different side of line xy with the center c; Let's say u. If both x and y are outside, then $\angle yux > \frac{\pi}{2}$. Thus, u is inside $disk(x, y)$, which contradicts that xy is a Gabriel edge. ■

Theorem 10: Assume two triangles $\triangle uvw$ and $\triangle xyz$ introduced to $LDel^{(k)}(S)$, $k \geq 1$, intersect, then either $disk(u, v, w)$ contains at least one of the nodes of $\{x, y, z\}$ or $disk(x, y, z)$ contains at least one of the nodes of $\{u, v, w\}$.

See the appendix for the proof. The above theorem guarantees that if two k-localized Delaunay triangles $\triangle uvw$ and $\triangle xyz$ intersect, then either $disk(u, v, w)$ or $disk(x, y, z)$ violates the Delaunay property by just considering the nodes $\{u, v, w, x, y, z\}$. We then show that $LDel^{(2)}(S)$ is a planar graph.

Theorem 11: $LDel^{(2)}(S)$ is a planar graph.

Proof: Notice that two Gabriel edges do not intersect. Then every intersection must involves a localized Delaunay triangle. Assume that an edge xy of $LDel^{(2)}(S)$ intersects a localized Delaunay triangle $\triangle uvw$. Edge xy is either a Gabriel edge or an edge of a localized Delaunay triangle, say $\triangle xyz$. If xy is a Gabriel edge, then Lemma 9 implies that either x or y is inside the $disk(u, v, w)$, say y. If xy is an edge of a localized Delaunay triangle $\triangle xyz$, then Theorem 10 implies that either x or y is inside the $disk(u, v, w)$, say y. The triangle inequality implies that

$$\|xu\| + \|yv\| < \|xy\| + \|uv\| \leq 2.$$

The existence of the 2-localized Delaunay triangle $\triangle uvw$ implies that $y \notin N_1(u) \cup N_1(v) \cup N_1(w)$. Thus, $\|yv\| > 1$, which implies that $\|xu\| < 1$. In other words, $x \in N_1(u)$. Consequently, $y \in N_2(u)$ because of the path yxu in the unit-disk graph $UDG(S)$, which contradicts to the existence of 2-localized Delaunay triangle $\triangle uvw$. The theorem follows. ■

We defined a sequence of localized Delaunay graphs $LDel^{(k)}(S)$, where $1 \leq k \leq n$. All graphs are t-spanner of the unit-disk graph with the following properties:

- $UDel(S) \subseteq LDel^{(k)}(S)$, for all $1 \leq k \leq n$;
- $LDel^{(k+1)}(S) \subseteq LDel^{(k)}(S)$, for all $1 \leq k \leq n$;
- $LDel^{(k)}(S)$ are planar graphs for all $2 \leq k \leq n$;
- $LDel^{(1)}(S)$ is not always planar.

D. $LDel^{(1)}(S)$ has thickness 2

In this subsection, we claim that $LDel^{(1)}(S)$ has thickness two, or in other words, its edges can be partitioned in two planar graphs. From Euler's formula, it follows that a simple planar graph with n nodes has at most $3n - 6$ edges, and therefore $LDel^{(1)}(S)$ has at most $6n$ edges. Due to space limitations, we omit the proof.

Theorem 12: Graph $LDel^{(1)}(S)$ has thickness 2.

V. LOCALIZED ALGORITHM

In this section, we study how to locally construct a planar t-spanner of $UDG(S)$. We assume that the identity of a node u can be represented by $O(\log n)$ bits and its location can be represented by $O(1)$ bits.

Although the graph $UDel(S)$ is a t-spanner for $UDG(S)$, we do not know how to construct it locally. We can construct $LDel^{(2)}(S)$, which is guaranteed to be a planar spanner of $UDel(S)$, but with a total communication cost of this approach is $O(m \log n)$ bits , where m is the number of edges in $UDG(S)$ and could be as large as $O(n^2)$. In order to reduce the total communication cost to $O(n \log n)$ bits, we do not construct $LDel^{(2)}(S)$, and instead we extract a planar graph $PLDel(S)$ out of $LDel^{(1)}(S)$.

A. Algorithm

Recall that $LDel^{(1)}(S)$ is not guaranteed to be a planar graph. We propose an algorithm that constructs $LDel^{(1)}(S)$ and then makes it a planar graph efficiently. The final graph still contains $UDel(S)$ as a subgraph. Thus, it is a t-spanner of the unit-disk graph $UDG(S)$.

In the following, the order of three nodes in a triangle is immaterial.

Algorithm 1: Localized Unit Delaunay Triangulation

1. Each wireless node u broadcasts its identity and location and listens to the messages from other nodes.
2. Assume that node u gathered the location information of $N_1(u)$. It computes the Delaunay triangulation $Del(N_1(u))$ of its 1-neighbors $N_1(u)$, including u itself.
3. For each edge uv of $Del(N_1(u))$, let $\triangle uvw$ and $\triangle uvz$ be two triangles incident on uv. Edge uv is a Gabriel edge if both angles $\angle uwv$ and $\angle uzv$ are less than $\pi/2$. Node u marks all *Gabriel edges* uv, which will never be deleted.
4. Each node u finds all triangles $\triangle uvw$ from $Del(N_1(u))$ such that all three edges of $\triangle uvw$ have length at most one unit. If angle $\angle wuv \geq \frac{\pi}{3}$, node u broadcasts a message proposal(u, v, w) to form a 1-localized Delaunay triangle $\triangle uvw$ in $LDel^{(1)}(V)$, and listens to the messages from other nodes.
5. When a node u receives a message proposal(u, v, w), u accepts the proposal of constructing $\triangle uvw$ if $\triangle uvw$ belongs to the Delaunay triangulation $Del(N_1(u))$ by broadcasting message accept(u, v, w); otherwise, it rejects the proposal by broadcasting message reject(u, v, w).
6. A node u adds the edges uv and uw to its set of incident edges if the triangle $\triangle uvw$ is in the Delaunay triangulation $Del(N_1(u))$ and both v and w have sent either accept(u, v, w) or proposal(u, v, w).

We first claim that the graph constructed by the above algorithm is $LDel^{(1)}(S)$. Indeed, for each triangle $\triangle uvw$ of $LDel^{(1)}(S)$, one of its interior angle is at least $\pi/3$ and $\triangle uvw$ is in $Del(N_1(u))$, $Del(N_1(v))$ and $Del(N_1(w))$. So one of the nodes amongst $\{u, v, w\}$ will broadcast the message proposal(u, v, w) to form a 1-localized Delaunay triangle $\triangle uvw$.

As $Del(N_1(u))$ is a planar graph, and a proposal is made only if $\angle wuv \geq \frac{\pi}{3}$, node u broadcasts at most 6 proposals. And each proposal is replied by at most two nodes. Therefore, the total communication cost is $O(n \log n)$ bits. The above algorithm also shows that $LDel^{(1)}(S)$ has $O(n)$ edges, which we know from Theorem 12. Putting together the arguments above, we have:

Theorem 13: Algorithm 1 constructs $LDel^{(1)}(S)$ with total communication cost $O(n \log n)$.

We then propose an algorithm to extract from $LDel^{(1)}(S)$ a planar subgraph.

Algorithm 2: Planarize $LDel^{(1)}(S)$

1. Each wireless node u broadcasts the Gabriel edges incident on u and the triangles $\triangle uvw$ of $LDel^{(1)}(S)$ and listens to the messages from other nodes.
2. Assume node u gathered the Gabriel edge and 1-local Delaunay triangles information of all nodes from $N_1(u)$. For two intersected triangles $\triangle uvw$ and $\triangle xyz$ known by u, node u removes the triangle $\triangle uvw$ if its circumcircle contains a node from $\{x, y, z\}$.
3. Each wireless node u broadcasts all the triangles incident on u which it has not removed in the previous step, and listens to the broadcasting by other nodes.
4. Node u keeps the edge uv in its set of incident edges if it is a Gabriel edge, or if there is a triangle $\triangle uvw$ such that u, v, and w have all announced they have not removed the triangle $\triangle uvw$ in Step 2.

We denote the graph extracted by the algorithm above by $PLDel(S)$. Note that any triangle of $LDel^{(1)}(S)$ not kept in the last step of the Planarization Algorithm is not a triangle of $LDel^{(2)}(S)$, and therefore $PLDel(S)$ is a supergraph of $LDel^{(2)}(S)$. Thus, by using Theorem 7, we have:

$$UDel(S) \subseteq LDel^{(2)}(S) \subseteq PLDel(S) \subseteq LDel^{(1)}(S)$$

Similar to the proof that $LDel^{(2)}(S)$ is a planar graph, we can show that our algorithm does generate a planar graph $PLDel(S)$. Due to space limitation, we omit the proof.

The total communication cost to construct the graph $PLDel(S)$ is a $O(\log n)$ times the number of edges of the graph $LDel^{(1)}(S)$, which by Theorem 12 is $O(n)$. Putting together all the arguments above and Theorem 6, we have:

Theorem 14: $PLDel(S)$ is planar $\frac{4\sqrt{3}}{9}\pi$-spanner of $UDG(S)$, and can be constructed with total communication cost $O(n \log n)$.

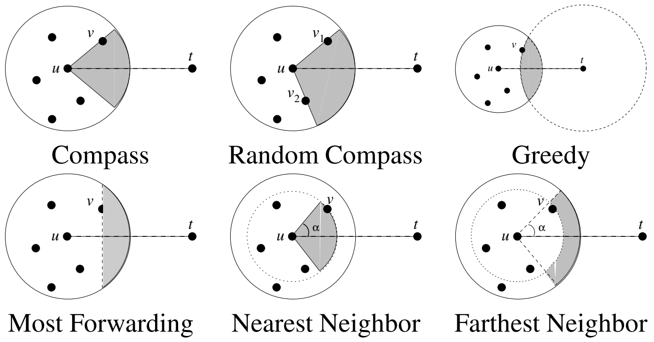
VI. ROUTING

We discuss how to route packets on the constructed graph. Recently, Bose and Morin [1] first proposed a localized routing algorithm that routes a packet from a source node s to a destination node t. Here a routing algorithm is localized if each relaying node decides to which node to forward the packet only based on the following information: the source node s, the destination node t, the current node u and all nodes of $N_k(u)$. We only use $k = 1$. Sometimes, the algorithm may use at most a constant number of bits of additional information. Their algorithm is based on the remarkable proof of Dobkin *et al.* [18] that the Delaunay triangulation is a t-spanner of the complete Euclidean graph. Bose and Morin [1] showed how to find another path locally with length no more than $\Pi_{dfs}(u, v)$. However their algorithm has a major deficiency by requiring the construction of the Delaunay triangulation and the Voronoi diagram of all wireless nodes, which could be very expensive in distributed computing.

Bose *et al.* [12] proposed another algorithm that routes the packets using the Gabriel graph to guarantee the delivery. Notice that the Gabriel graph is a subgraph of $PLDel(S)$. Thus, if we apply the routing method proposed in [12] on the newly proposed planar graph $PLDel(S)$, we expect to achieve better performance because $PLDel(S)$ is denser than the Gabriel graph (but still with $O(n)$ edges). The constructed local Delaunay triangulation not only guarantees that the length of the shortest path connecting any two wireless nodes is at most a constant factor of the minimum in the unit-disk graph, it also guarantees that the energy consumed by the path is also minimum, as it includes the Gabriel graph (see [29], [21]). Moreover, because the constructed topology is planar, then a localized routing algorithm using the right hand rule guarantees the delivery of the packets from source node to the destination node.

We study the following routing algorithms on the graphs proposed in this paper.

Compass Routing Let t be the destination node. Current node u finds the next relay node v such that the angle $\angle vut$ is the smallest among all neighbors of u in a given topology. See[14].



Compass Random Compass Greedy

Most Forwarding Nearest Neighbor Farthest Neighbor

Fig. 5. Shaded area is empty and v is next node.

Random Compass Routing Let u be the current node and t be the destination node. Let v_1 be the node on the above of line ut such that $\angle v_1ut$ is the smallest among all such neighbors of u. Similarly, we define v_2 to be nodes below line ut that minimizes the angle $\angle v_2ut$. Then node u randomly choose v_1 or v_2 to forward the packet. See[14].

Greedy Routing Let t be the destination node. Current node u finds the next relay node v such that the distance $\|vt\|$ is the smallest among all neighbors of u in a given topology. See [12].

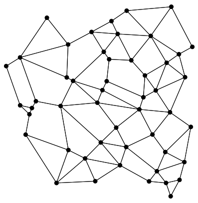
Most Forwarding Routing (MFR) Current node u finds the next relay node v such that $\|v't\|$ is the smallest among all neighbors of u in a given topology, where v' is the projection of v on segment ut. See [11].

Nearest Neighbor Routing (NN) Given a parameter angle α, node u finds the nearest node v as forwarding node among all neighbors of u in a given topology such that $\angle vut \leq \alpha$.

Farthest Neighbor Routing (FN) Given a parameter angle α, node u finds the farthest node v as forwarding node among all neighbors of u in a given topology such that $\angle vut \leq \alpha$.

Notice that it is shown in [12], [14] that the compass routing, random compass routing and the greedy routing guarantee to deliver the packets from the source to the destination if Delaunay triangulation is used as network topology. They proved this by showing that the distance from the selected forwarding node v to the destination node t is less than the distance from current node u to t. However, the same proof cannot be carried over when the network topology is Yao graph, Gabriel graph, relative neighborhood graph, and the localized Delaunay triangulation. When the underlying network topology is a planar graph, the right hand rule is often used to guarantee the packet delivery after simple localized routing heuristics fail [12], [11], [13].

We present our experimental results of various routing methods on different network topologies. Figure 6 illustrates some network topologies discussed in this paper. Recall that Gabriel graph, relative neighborhood graph, Delaunay triangulation, $LDel^{(2)}(S)$, and $PLDel(S)$ are always planar graphs. The Yao structure, Delaunay triangulation, $LDel^{(2)}(S)$, and $PLDel(S)$ are always a t-spanner of the unit-disk graph. We use integer parameter $k = 8$ in constructing the Yao graph. In the experimental results presented here, we choose total $n = 50$ wireless nodes



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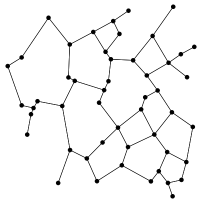

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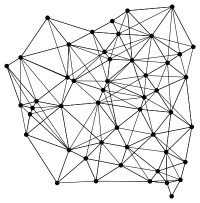

Yao