Homework Assignment 1
CS 330 Discrete Structures
Summer Semester, 2016

1. (5pts) User De Morgan’s Law to find the negation of the following statements.

   - Chris knows Java and C.
     - Chris does not know Java or C.
   - Willis tower is taller than Trump tower, and it is also taller than John Hancock tower.
     - Willis tower is shorter than either Trump tower or John Hancock tower.

2. (5pts) Use two methods (1. logical deduction, 2. truth table) to show that \((\neg p \land (p \to q)) \to \neg p\) is a tautology.

   (a) Logical deduction using equivalences

\[
(-p \land (p \to q)) \to \neg p \equiv (-p \land \neg p \lor q) \to \neg p
\equiv \left(\neg p \land \neg p \lor (\neg p \land q)\right) \to \neg p
\equiv (\neg p \lor (\neg p \land q)) \to \neg p
\equiv \neg p \to \neg p
\equiv \neg (\neg p) \lor \neg p
\equiv p \lor \neg p
\equiv T
\]
(b) Using a truth table

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<thead>
<tr>
<th>p</th>
<th>q</th>
<th>p \rightarrow q</th>
<th>\neg p \land (p \rightarrow q)</th>
<th>(\neg p \land (p \rightarrow q)) \rightarrow \neg p</th>
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</thead>
<tbody>
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3. (5pts) Evaluate the truth value of the following statements and justify the reason.

- If CS330 summer 2016 is open in Illinois Tech., the instructor of it will be female.
  - CS330 is open in summer 2016, but the instructor is male. This corresponds to \( T \rightarrow F \), therefore is false.
- If the instructor of CS330 summer 2016 is female, it will not be open.
  - The instructor is male (false pre-condition). An implication with false pre-condition is always true, therefore this is true.
- A student will get an A in CS330 if and only if he/she has the highest weighted total in the class at the end of the semester.
  - This sentences is equivalent to
    \[
    \text{(Get an A } \rightarrow \text{ Highest score)} \land (\text{Highest score } \rightarrow \text{ Get an A})
    \]
    The second implication is true, but the first implication is false. Therefore, the entire statement is false.

4. (5pts) Negate the following propositions.

- \( \exists \varepsilon, \exists N, \forall n : (n < N \land \varepsilon > 0 \land N < 0) \rightarrow (|\frac{1}{n}| - \frac{1}{n} < \varepsilon) \)
  - \[
  \neg(\exists \varepsilon, \exists N, \forall n : (n < N \land \varepsilon > 0 \land N < 0) \rightarrow (|\frac{1}{n}| - \frac{1}{n} < \varepsilon))
  \]
  \[
  \equiv \forall \varepsilon, \forall N, \exists n : (n < N \land \varepsilon > 0 \land N < 0) \land (|\frac{1}{n}| - \frac{1}{n} \leq \varepsilon)
  \]
- \( \exists \varepsilon > 0, \exists N < 0, \forall n > N : |\frac{1}{n}| - \frac{1}{n} < \varepsilon \)
  - \[
  \neg(\exists \varepsilon > 0, \exists N < 0, \forall n > N : |\frac{1}{n}| - \frac{1}{n} < \varepsilon)
  \]
  \[
  \equiv \forall \varepsilon > 0, \forall N < 0, \exists n > N : (|\frac{1}{n}| - \frac{1}{n} \leq \varepsilon)
  \]
  \[
  \equiv \forall \varepsilon > 0, \forall N < 0, \exists n > N : (|\frac{1}{n}| - \frac{1}{n} \geq \varepsilon)
  \]
5. (10pts) Let $P(x)$ be the statement ‘$x$ is a student in CS330’, $Q(x)$ be the statement ‘$x$ is in Illinois Tech.’, $R(x)$ be the statement ‘$x$ is an instructor’, and $S(x)$ be the statement ‘$x$ wants to play Starcraft’. Express the following statements using quantifiers, logical connectives and $P(x), Q(x), R(x), S(x)$.

(a) No instructor wants to play Starcraft.

\[-(\exists x : R(x) \land S(x)) \equiv \forall x : R(x) \rightarrow \neg S(x)\]

(b) All students want to play Starcraft.

\[\forall x : P(x) \rightarrow S(x)\]

(c) All students in CS330 are in Illinois Tech.

\[\forall x : P(x) \rightarrow Q(x)\]

(d) No instructor is not in Illinois Tech.

\[-(\exists x : R(x) \land \neg Q(x)) \equiv \forall x : R(x) \rightarrow Q(x)\]

Can we deduce an instructor is not a student of CS330 from any combination of above propositions?

Essentially, we need to deduce $\forall x : R(x) \rightarrow \neg P(x)$. From (a), we know $\forall x : R(x) \rightarrow \neg S(x)$, and from (b), we know $\forall x : P(x) \rightarrow S(x) \equiv \forall x : \neg S(x) \rightarrow \neg P(x)$ (contrapositive). Combining these two, we have

\[\forall x : R(x) \rightarrow \neg S(x) \rightarrow \neg P(x) \Rightarrow \forall x : R(x) \rightarrow \neg P(x)\]

6. (10pts) Use predicates, quantifiers, logical connectives and mathematical operations to express the statement ‘There is a positive integer that is not a perfect square’.

\[x, y’s \ domain \ is \ Z^+.

\[\exists x, \forall y : x \neq y^2\]

7. (10pts) Prove by contradiction that at least one Illinois resident’s salary is greater than or equal to the average salary of Illinois.

The original statement is ‘$\exists x : x’s \ salary \ \geq \ average \ salary’$. Let’s assume this if false. Then, we are assuming

\[-(\exists x : x’s \ salary \ \geq \ average \ salary) \equiv \forall x : x’s \ salary < average \ salary\]
The definition of the average salary is total salary divided by number of people. Therefore, average salary multiplied by the number of people should be the total salary. However, our assumption says everyone’s salary is lower than the average. That is,

$$\sum x\text{'s salary} < \text{average salary} \times \# \text{ of people} = \text{Total salary}$$

However, \(\sum x\text{'s salary}\) is also total salary in Illinois, and we have an inequality ‘Total salary < Total salary’, which is a contradiction.

8. (10pts) Prove or disprove:

$$\left((\exists x : P(x)) \land (\exists x : Q(x))\right) \iff \left(\exists x : (P(x) \land Q(x))\right)$$

A counterexample exists: \(x\)’s domain is \(\mathbb{R}\), \(P(x) = x > 0\), \(Q(x) = x < 0\). Then,

$$\left(\exists x : P(x)\right) \land \left(\exists x : Q(x)\right) = T \land T = T$$

and

$$\exists x : (P(x) \land Q(x)) = F$$

9. (10pts) Prove by contradiction that, if one rolls a die for 25 times, at least 5 of them must show the same number.

The statement is ‘rolls 25 times \(\rightarrow\) at least 5 of them are same numbers’. We are assuming this is false, so we are assuming ‘rolls 25 times \(\rightarrow\) at most 4 of them are same numbers’.

At most 4 of them are same numbers, then at most we will have 4 1’s, 4 2’s, 4 3’s, 4 4’s, 4 5’s, 4 6’s as the outcome, then this is only 24 times. There is no way we can roll a die for 25 times, which is a contradiction.

10. (10pts) Suppose we have a statement \(S_n\) and we know the following facts.

(a) \(S_1\) is true.
(b) If \(S_n\) is true, then \(S_{n-1}\) is true as well.
(c) If \(S_n\) is true, then \(S_{2n}\) is true as well.

Prove by induction that \(S_n\) is true for all integers \(n \geq 1\).

**Base case:** (a) directly proves the base case.

**Inductive hypothesis:** assume \(S_k\) is true.

**Inductive step & proof:** we want to prove \(S_k \rightarrow S_{k+1}\). Because of (c), \(S_k \rightarrow S_{2k}\), and because of repeated application of (b), \(S_{2k} \rightarrow S_{2k-1} \rightarrow S_{2k-2} \rightarrow S_{2k-3} \rightarrow \cdots \rightarrow S_{k+2} \rightarrow S_{k+1}\). This is done by applying (b) for \(k - 1\) times. Then, by combining the above two, we have \(S_k \rightarrow S_{2k} \rightarrow S_{k+1}\), which proves \(S_k \rightarrow S_{k+1}\).
11. (10pts) Prove by induction that
\[ \forall n \geq 1 : 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + (n - 1) \cdot (n - 1)! + n \cdot n! = (n + 1)! - 1 \]

**Base case:** when \( n = 1 \), \( LHS = 1 \cdot 1! = 1 \), \( RHS = (1 + 1)! - 1 = 1 \), therefore when \( n = 1 \), the equality holds.

**Inductive hypothesis:** when \( n = k \), assume the equality holds. That is, assume
\[ 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + (k - 1) \cdot (k - 1)! + k \cdot k! = (k + 1)! - 1 \]

**Inductive step and proof:** We want to prove \( LHS = RHS \) when \( n = k + 1 \) based on the above assumption. When \( n = k + 1 \)
\[
LHS = \left( 1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! \right) + (k + 1) \cdot (k + 1)!
\]
\[
= \left( (k + 1)! - 1 \right) + (k + 1) \cdot (k + 1)! \quad \text{(substitution by following the assumption)}
\]
\[
= (k + 1)! \cdot (k + 2) - 1
\]
\[
= (k + 2)! - 1
\]
\[
= RHS
\]

12. (30pts) Prove by induction that
\[ \forall n \geq 6 : \left( \frac{n}{3} \right)^n < n! < \left( \frac{n}{2} \right)^n \]

with the hint: \( 2 < (1 + \frac{1}{n})^n < 3 \) for all \( n \geq 0 \).

**Domain of \( n \):** is the integers \( \geq 5 \).

**First side \( \left( \frac{n}{3} \right)^n < n! \):**

**Base case:** when \( n = 6 \), \( LHS = 2^6 = 64 \), \( RHS = 6! = 720 \), therefore \( LHS < RHS \).

**Inductive hypothesis:** when \( n = k > 6 \), assume the inequality holds. That is, assume
\[ \left( \frac{k}{3} \right)^k < k! \]

**Inductive step & proof:** when \( n = k + 1 > 6 \),
\[
LHS = \left( \frac{k + 1}{3} \right)^{k+1}
\]

It seems hard to find out the common parts with the hypothesis from the LHS, so let’s try at the RHS.
\[
RHS = (k + 1)! = (k + 1) \cdot k!
\]

According to the hypothesis, \( k! > \left( \frac{k}{3} \right)^k \), therefore we should have:
\[
RHS = (k + 1) \cdot k! > (k + 1) \cdot \left( \frac{k}{3} \right)^k
\]
We want to show that \((k + 1)(k/3)^k \geq ((k + 1)/3)^{k+1}\) so that \(RHS > ((k + 1)/3)^{k+1} = LHS\).

\[
\left(\frac{k+1}{3}\right)^{k+1} \leq (k + 1)\left(\frac{k}{3}\right)^k
\]

\[
= \frac{1}{3} (k+1)^k \leq \left(\frac{k}{3}\right)^k
\]

\[
= \frac{1}{3}(k+1)^k \leq k^k
\]

\[
= (k+1)^k \leq 3k^k
\]

\[
= \left(1 + \frac{1}{k}\right)^k \leq 3
\]

\[
= T \quad \text{according to the hint}
\]

Therefore, \( \left(\frac{k+1}{3}\right)^{k+1} \) is indeed less than or equal to \((k + 1)\left(\frac{k}{3}\right)^k\), and therefore \(LHS < RHS\) when \(n = k + 1\). This completes the proof for the first side.

Second side \(n! < \left(\frac{n}{2}\right)^n\):

**Base case:** when \(n = 6\), \(LHS = 6! = 720\), \(RHS = 3^6 = 729\), therefore \(LHS < RHS\).

**Inductive hypothesis:** when \(n = k > 6\), assume the inequality holds. That is, assume

\[
k! < \left(\frac{k}{2}\right)^k
\]

**Inductive step & proof:** when \(n = k + 1 > 6\),

\[
LHS = (k + 1)! = (k + 1)k! < (k + 1)\left(\frac{k}{2}\right)^k
\]

We want to show that \((k + 1)\left(\frac{k}{2}\right)^k \leq \left(\frac{k+1}{2}\right)^{k+1}\) so that \(LHS < \left(\frac{k+1}{2}\right)^{k+1} = RHS\).

\[
(k + 1)\left(\frac{k}{2}\right)^k \leq \left(\frac{k+1}{2}\right)^{k+1}
\]

\[
= \left(\frac{k}{2}\right)^k \leq \frac{1}{2} \left(\frac{k+1}{2}\right)^k
\]

\[
= k^k \leq \frac{1}{2} (k+1)^k
\]

\[
= 2k^k \leq (k+1)^k
\]

\[
= 2 \leq \left(1 + \frac{1}{k}\right)^k
\]

\[
= T \quad \text{according to the hint}
\]

Therefore, \((k + 1)\left(\frac{k}{2}\right)^k\) is indeed less than or equal to \(\left(\frac{k+1}{2}\right)^{k+1}\), and therefore \(LHS < RHS\) when \(n = k + 1\). This completes the proof for the second side.

In conclusion, the original inequality holds when \(n \geq 6\).