Homework Assignment 3
CS 330 Discrete Structures
Summer Semester, 2016

Name:
A-ID:

Due: 9:30am June 3, 2016

Instruction: Leave the notations as they are without calculating the values.

1. (15pts) I made 50 problems for Exam 1, and 15 problems will be chosen as the Exam 1 next week.
   • The CS coordinator asks me to send him/her my choice before Friday. How many different choices can I make?
     I need to choose 15 from 50, so it’ll be \( \binom{50}{15} \).
   • The CS coordinator wants to print out the 15 problems as Exam 1 in certain order. How many different ways are there to print them out?
     The coordinator may choose arbitrary order among 15 problems, so he/she has 15! different ways to print them.
   • If I simply choose and print out the problems my self, how many different ways are there to do that?
     If I do everything on my own, then it will be the combination of the above two: \( \binom{50}{15} \cdot 15! = P(50, 15) \).

2. (20pts) A fair coin is flipped 20 times, and the head or tail will come up equally likely. Then, how many possible outcomes
   • contain exactly three heads?
     3 out of 20 times may be head, therefore it is \( \binom{20}{3} \).
   • contain at least three heads?
     The negation of this is ‘at most two heads’, which is ‘0 head’, ‘1 head’, or ‘2 heads’. The number of corresponding outcomes is \( \binom{20}{0} + \binom{20}{1} + \binom{20}{2} \). The number of entire possible outcomes is \( 2^n \), therefore the answer will be \( 2^n - \binom{20}{0} - \binom{20}{1} - \binom{20}{2} \).
   • contain the same number of heads and tails?
     This means 10 heads and 10 tails. Therefore, it will be \( \binom{20}{10} \). Note that it is not \( \binom{20}{10} / 2 \) because there is no over-counting. For example, ‘first 10 times showing heads and the last 10 times showing tails’ is indeed different from ‘first 10 times showing tails and the last 10 times showing heads’.
   • contain 5 heads and 15 tails?
     It is either \( \binom{20}{5} \) or \( \binom{20}{15} \).
3. (20pts) Suppose we are trying to choose two students among 2n students. Give two different ways to choose 2 students to show that
\[
\binom{2n}{2} = 2 \binom{n}{2} + n^2
\]
In the LHS, the formula implies it is the number of ways to choose 2 students from 2n.
We can interpret the RHS as follows. Divide 2n students into two: G1 having n students and G2 having the other n students. In order to choose 2 students, we can do one of the three ways below:

(a) Choose 2 students from G1: \( \binom{n}{2} \)
(b) Choose 2 students from G2: \( \binom{n}{2} \)
(c) Choose 1 from G1 and another 1 from G2: \( \binom{n}{1} \cdot \binom{n}{1} \)

The addition of all three will be equal to RHS. Essentially the two are doing the same counting, therefore LHS and RHS should be equal to each other.

4. (20pts)

- A parking lot has a row of n parking spaces, and k compact cars (occupying one space) arrived at once for parking now. How many different arrangements can we make for the empty parking spaces? and how many different arrangements can we make for the vehicles?
  
  Since empty spaces are not distinguishable from each other, we only need to choose \( n-k \) empty spaces out of n parking spaces: \( \binom{n}{n-k} \).
  
  For the vehicles, the order of the cars matters, therefore it is \( P(n, k) = \binom{n}{k} \cdot k! \).

- An SUV occupies two adjacent spaces. If k compact cars arrived, then how many different arrangements for the empty parking spaces are there in which an SUV cannot park? That is, how many arrangements of empty spaces do not have two adjacent empty spaces?
  
  (Hint: Imagine k cars already parked adjacently, and how many ways are there to insert \( n-k \) empty spaces?)
  
  So k cars are now parked next to each other, and we are trying to insert \( n-k \) empty spaces among them. Because we do not want SUV to be parked, at most one space can be inserted between any car or at each end. Spots between the cars is \( k-1 \) for k cars, and we have two spots at each end, therefore we have \( k+1 \) space to choose for inserting \( n-k \) empty spaces: \( \binom{k+1}{n-k} \).

5. (10pts) There is a deck of 52 cards

- If we evenly distribute the 52 cards to 4 players A,B,C, and D (i.e., 13 cards for each player), how many different ways are there to distribute the cards?
  
  Since the order of the cards within each player does not matter, it is: \( \frac{\binom{52}{13} \cdot \binom{39}{13} \cdot \binom{26}{13} \cdot \binom{13}{13}}{4!} \).

- If we evenly divide the 52 cards to 4 groups (i.e., 13 cards each group), how many different ways are there to divide the cards?
  
  There is over-counting in the answer \( \binom{52}{13} \cdot \binom{39}{13} \cdot \binom{26}{13} \cdot \binom{13}{13} \) as we discussed in the lecture. Same set of cards being chosen into 1st group or 2nd group or 3rd group or 4th group does not really make any difference but all are counted as different cases in that answer. Therefore, we divide it by 4! to get rid of it since every 4! cases in that answer in fact corresponds to only one case in this problem. So the answer is \( \frac{\binom{52}{13} \cdot \binom{39}{13} \cdot \binom{26}{13} \cdot \binom{13}{13}}{4!} \).
6. (10pts) Find the coefficient of $x^2$ in each of the following polynomials.

- $(1 + 2x)^{2016}$
  According to the formula, the term containing $x^2$ is $\binom{2016}{2} \cdot 1^{2016-2} \cdot (2x)^2 = 4\binom{2016}{2}x^2$. Therefore, the coefficient is $4\binom{2016}{2}$.

- $(1 - 2013x)^{2n+1}$
  Similarly, the term containing $x^2$ is $\binom{2n+1}{2} \cdot 1^{2n+1-2} \cdot (-2013x)^2 = 2013^2 \cdot \binom{2n+1}{2}x^2$. Therefore, the coefficient is $2013^2 \cdot \binom{2n+1}{2}$.

7. (10pts) Simplify the following sum, and show how you did so by using the binomial coefficients.

$$16x^8 + 24x^7 + 36x^6 + 54x^5 + 81x^4$$

It is easy to verify that this is the expanded version of $(2x^2 + 3x)^4$.

8. (15pts) Use Pascal’s identity to prove that

$$\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r}$$

(Hint: proof by induction will work)

The original equation is

$$\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \binom{n+3}{3} + \cdots + \binom{n+r}{r} = \binom{n+r+1}{r}$$

We prove this by induction on $r$.

**Base case:** When $r = 0$, it is just $\binom{n}{0} = \binom{n+1}{0}$ which is obviously true.

**Inductive Hypothesis:** When $r = x$, let’s assume the equality holds. That is,

$$\sum_{k=0}^{x} \binom{n+k}{k} = \binom{n+x+1}{x}$$

**Inductive Step:** When $r = x + 1$, we have

$$LHS = \sum_{k=0}^{x+1} \binom{n+k}{k} = \left( \sum_{k=0}^{x} \right) + \binom{n+x+1}{x+1}$$

$$= \binom{n+x+1}{x} + \binom{n+x+1}{x+1}$$

$$= \binom{n+x+2}{x+1} = RHS$$

Therefore, the equality holds for any $r \geq 0$. 
9. (20pts) I am going to give an A+ to anyone beating me in a 1-on-1 game in Starcraft 2. I only play Zerg, and I have the following records:

(a) 50% to win against a Terran player
(b) 60% to win against a Protoss player
(c) 75% to win against Zerg player

Suppose you are trying to quickly get an A+ without studying by beating me up in Starcraft 2, but you did not have access to my record in advance and selected ‘random’. Consequently, you will randomly get assigned Terran, Protoss, or Zerg with the probability \( \frac{1}{3} \) each when the game starts. Now, what is your total probability to win the game? Show your work.

So we have three cases:

- You are assigned Terran with \( \frac{1}{3} \) probability, and then your probability to win is 50%.
- You are assigned Protoss with \( \frac{1}{3} \) probability, and then your probability to win is 40%.
- You are assigned Zerg with \( \frac{1}{3} \) probability, and then your probability to win is 25%.

Therefore, your final probability to win is, by the rule of product and sum:

\[
\Pr[\text{win}] = \Pr[\text{win} \land \text{Terran}] + \Pr[\text{win} \land \text{Protoss}] + \Pr[\text{win} \land \text{Zerg}]
\]
\[
= \Pr[\text{win} | \text{Terran}] \cdot \Pr[\text{Terran}] + \Pr[\text{win} | \text{Protoss}] \cdot \Pr[\text{Protoss}] + \Pr[\text{win} | \text{Zerg}] \cdot \Pr[\text{Zerg}]
\]
\[
= \frac{1}{3} \cdot 50\% + \frac{1}{3} \cdot 40\% + \frac{1}{3} \cdot 25\%
\]

At the end of the semester, you notice Alice got an A+ even though she didn’t come to classes at all. Statistics of CS Department indicate

(a) The overall probability that a student gets an A+ in CS330 is 10%.
(b) The probability a student never coming to classes gets an A+ is 2%.

You also heard Alice saying ‘With 80%, I’ll just give up and try to beat him up in Starcraft 2, and with 20% I’ll study on my own.’

- What is the probability that Alice earned A+ by studying on her own? Show your work.

\[
\Pr[\text{on her own}|\text{A+}] = \frac{\Pr[\text{A+} | \text{on her own}] \cdot \Pr[\text{on her own}]}{\Pr[\text{A+}]}
\]

According to the CS department’s statistics, \( \Pr[\text{A+} | \text{on her own}] = 2\% \) because Alice never came to classes, and Alice said \( \Pr[\text{on her own}] = 20\% \). Besides, the statistics also suggests \( \Pr[\text{A+}] = 10\% \). Plugging in all the probabilities into the above formula will give the final answer.
10. (20pts) We have a magic coin. Initially, the chances of head and tail are fair (\( \frac{1}{2} \) each), but every time we toss it, the probability of heads will decrease by a multiplicative factor of \( \frac{1}{2} \) so that at the \( i \)-th toss, the probability of head will be \( (\frac{1}{2})^i \). We toss this coin for \( n \) times.

- What is the probability that all \( n \) tosses are heads? Show your work.
  
  The probability that the coin shows head is \( \frac{1}{2} \) at the first toss, \( (\frac{1}{2})^2 \) at the second toss, \( (\frac{1}{2})^3 \) at the third toss, \( \ldots \), and \( (\frac{1}{2})^n \) at the \( n \)-th toss. According to the rule of product, the final probability will be

  \[
  (\frac{1}{2}) \cdot (\frac{1}{2})^2 \cdots (\frac{1}{2})^n = \prod_{i=1}^{n} (\frac{1}{2})^i = (\frac{1}{2})^{\sum_{i=1}^{n} i}
  \]

  * It is not necessary to abbreviate the expression with \( \prod \) or \( \sum \).

- What is the expected number of heads we will get? Show your work.

  Let’s say \( I_i = 1 \) if the coin shows head at the \( i \)-th toss and 0 otherwise. Essentially, we are trying to calculate

  \[
  E[I_1 + I_2 + I_3 + \cdots + I_{n-1} + I_n] = E[I_1] + E[I_2] + \cdots + E[I_n]
  \]

  According to the definition of the expected value, for any \( i \):

  \[
  E[I_i] = 1 \cdot \Pr[I_i = 1] + 0 \cdot \Pr[I_i = 0] = (\frac{1}{2})^i
  \]

  Therefore, the above total expected value is

  \[
  E[I_1] + E[I_2] + \cdots + E[I_n] = \frac{1}{2} + (\frac{1}{2})^2 + (\frac{1}{2})^3 + \cdots + (\frac{1}{2})^n
  \]

  This is the sum of a geometric sequence and there is a corresponding formula, but it is not necessary to use it to calculate the final result in this course.