Name:  
A-ID:  

Due: 8:50am June 8, 2016

1. (10pts) \( f(x) \) is \( \Theta(g(x)) \). Disprove the following statement.

   It is always true that \( t(f(x)) \) is \( \Theta(t(g(x))) \)

   One simple counterexample disproves the statement: \( t(x) = 2^x, f(x) = x, g(x) = 2x \). In this case, even though \( f(x) = \Theta(g(x)) \),
   
   \[ 2^f(x) = 2^x \neq \Theta(2^{g(x)}) = \Theta(2^{2x}) = \Theta(4^x) \]

2. (30pts) Sort the following functions according to their growth rates (from the slowest to the fastest).
   You do NOT need to justify the answers.

   \( \left(\frac{3}{2}\right)^n, n^3, n \log_2(n^n), 2^n, 2^{(2n)}, 4^{\log_2 n}, (\log_2 n)^2, n! \)

   \( (\log_2 n)^2, 4^{\log_2 n} = n^{\log_2 4} = n^2, n \log_2 (n^n) = n^2 \log_2 n, n^3, \left(\frac{3}{2}\right)^n, 2^n, 2^{(2n)} = 4^n, n! \)

3. (15pts) Prove that

   \( f(x) = 3x \) is \( O(x) \). That is, find out the \( x_0 \) and \( c \) which satisfies \( \forall x > x_0 : f(x) \leq cx \) (this is the definition of Big-Oh).

   \[ 3x \leq 4x \text{ when } x > 0 \]
   
   therefore \( c = 4 \), and \( x_0 \) can be any positive number.

   \( f(x) = 4x \) is \( o(x^2) \). That is, find out the \( x_0 \) according to the definition of Little-Oh notation.

   \[ \forall c > 0 : x < cx^2 \text{ when } x > \frac{1}{c} \]

   therefore for any \( c > 0 \), \( x_0 \) can be any number that is greater than \( \frac{1}{c} \).
• \( f(x) = 3x \) is \( \Theta(x) \). Similarly, refer to the definition of Big-Theta notation and find out \( x_0, c_1 \) and \( c_2 \).

\[
x \leq 3x \leq 5x \text{ when } x > 0
\]

therefore \( c_1 = 1, c_2 = 5 \) and \( x_0 \) can be any positive number.

Note that many other \( c \)'s can be used to prove them too. Those are just examples.

4. (10pts) It is one of the properties of logarithm that:

\[
\forall a > 0, b > 0, c > 0 : \log_a b = \frac{\log_c b}{\log_c a}
\]

Explain why the base of the logarithm (i.e., \( a \) in \( \log_a x \)) does not affect the growth rate of \( f(x) = \log_a x \).

\[
\log_a x = \frac{\log_b x}{\log_b a}
\]

Therefore, the change in the base only leads to the change by a constant factor, which does not affect the growth rate of the function.

5. (10pts) On the other hand, explain why \( f(x) = a^x \) grows faster than \( g(x) = b^x \) when \( a > b > 1 \).

(Hint: try getting the \( \lim_{x \to +\infty} \) of their proportion)

\[
\lim_{x \to +\infty} \frac{a^x}{b^x} = \left( \frac{a}{b} \right)^x = +\infty
\]

therefore \( a^x \) grows faster than \( b^x \).

6. (20pts) Given the following algorithm.

```
1: sum ← 0
2: for i ← 1 to n do
3:   for j ← 1 to n do
4:     if i < j then
5:       sum ← sum + 1
6:     end if
7:   end for
8: end for
9: return sum
```

• Analyze the time complexity of the algorithm.

The high-order component in the time complexity will be determined by line 4 and line 5. Line 4 is executed for \( \Theta(n^2) \) times and line 5 is executed for \( O(n^2) \) times. In overall, the time complexity will be \( \Theta(n^2) \).

• What is the value of \( sum \) at the end?

The algorithm adds 1 if \( i < j \) in the nested for loops. There are \( \binom{n}{2} \) such pairs, therefore \( sum = \binom{n}{2} \).