1. (10pts) Sort the following integers by using the ‘merge sort’, and show its procedure step by step similar to Section 1.2 in Lecture note 5.

14, 1, 4, 8, 2, 9, 10, 11, 3, 7, 15, 5, 13, 6, 12

Start dividing.

14, 1, 4, 8, 2, 9, 10, 11 3, 7, 15, 5, 13, 6, 12
14, 1, 4, 8 2, 9, 10, 11 3, 7, 15, 5 13, 6, 12
14, 1 4, 8 2, 9 10, 11 3 7 15 5 13 6 12

Start merging.

1, 14 4, 8 2, 9 10, 11 3, 7 5, 15 6, 13 12
1, 4, 8, 14 2, 9, 10, 11 3, 5, 7, 15 6, 12, 13
1, 2, 4, 8, 9, 10, 11, 14 3, 5, 6, 7, 12, 13, 15
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15

2. (20pts) Get the Big-Theta notations of the following recurrences by using Master Theorem.

- \( T(n) = 3T\left(\frac{n}{3}\right) + n \)
  \[ a = 3, b = 3 \Rightarrow \frac{af(n/b)}{f(n)} = \frac{n}{n} = 1 \Rightarrow T(n) = \Theta(n \log n) \]

- \( T(n) = 2T\left(\frac{n}{4}\right) + n^2 \)
  \[ a = 2, b = 4 \Rightarrow \frac{af(n/b)}{f(n)} = \frac{n^2/8}{n^2} = \frac{1}{8} < 1 \Rightarrow T(n) = \Theta(n^2) \]
• \( T(n) = 4T\left(\frac{n}{2}\right) + \log n \)

\[
a = 4, b = 2 \Rightarrow \frac{af(n/b)}{f(n)} = \frac{4\log\left(\frac{n}{2}\right)}{\log n} = \frac{4(\log n - \log 2)}{\log n} > 1 \Rightarrow T(n) = \Theta(n^{\log_2 4}) = \Theta(n^2)
\]

3. (30pts) You are riding a motorcycle from Chicago to New York, and you found there are \( n \) gas stations on the route. Because the tank of your motorcycle is small, you need to refuel the bike frequently, and you can pass at most 1 gas station without refuelling. Let’s see how many refuel strategies you can have throughout your trip to New York.

• Let \( S(n) \) be the total number of refuel strategies. Identify the recurrence of \( S(n) \).

From the start point,

(a) If you choose to refuel at the 1st station, the number of refuelling strategies you have after then is \( S(n-1) \) since the number of the remaining gas stations is \( n - 1 \).

(b) If you choose to skip the 1st station, then you have to refuel at the 2nd station. After then, the number of refuelling strategies you will have is \( S(n-2) \).

Since the first step is different, the above two different cases lead to different strategies that are independent. Therefore, \( S(n) = S(n-1) + S(n-2) \).

• What is the annihilator of \( S(n) \)?

\[
(E^2 - E - 1) = (E - \frac{1+\sqrt{5}}{2})(E - \frac{1-\sqrt{5}}{2})
\]

• What is the exact solution of \( S(n) \)?

The general solution is \( S(n) = a\left(\frac{1+\sqrt{5}}{2}\right)^n + b\left(\frac{1-\sqrt{5}}{2}\right)^n \). As I announced by email, \( S(0) = 1, S(1) = 2, S(2) = 3, \cdots \) are the initial conditions. Using them to solve the linear system will give \( a \) and \( b \), which are

\[
a = \frac{1}{2}(1 + \frac{3}{\sqrt{5}}), \quad b = \frac{1}{2}(1 - \frac{3}{\sqrt{5}})
\]

4. (20pts) What is the general solution of

\[
a_n = 3a_{n-1} - 2a_{n-2} + F(n)
\]

The annihilator for the homogeneous part is \((E - 1)(E - 2)\)

• if \( F(n) = 3^n \)?

The annihilator for \( F(n) \) is \((E - 3)\), which makes the total annihilator as

\[(E - 1)(E - 2)(E - 3)\]

Therefore the general solution is \( F(n) = a + b2^n + c3^n \).

• if \( F(n) = 2^n \)?

The annihilator for \( F(n) \) is \((E - 2)\), which makes the total annihilator as

\[(E - 1)(E - 2)^2\]

Therefore the general solution is \( F(n) = a + (bn + c)2^n \).
5. (20pts) Prove by induction on $n$ that the operator $(E-a)^{n+1}$ annihilates any sequence $\langle P^{(n)}(i)a^i \rangle$ where $P^{(n)}(i)$ is a polynomial in $i$ of degree $n$.

**Base case:** when $n = 1$,

$$(E-a)^2(\langle \alpha i + \beta \rangle a^i) = (E-a)(\langle \alpha(i+1) + \beta \rangle a^{i+1} - \langle \alpha i + \beta \rangle a^{i+1}) = (E-a)(\alpha a^{i+1}) = 0$$

Therefore, $(E-a)^{n+1}$ does annihilate $\langle P^{(n)}(i)a^i \rangle$ when $n = 1$.

**Inductive hypothesis:** assume $(E-a)^{n+1}$ annihilate $\langle P^{(n)}(i)a^i \rangle$ when $n = k$.

**Inductive step:** when $n = k + 1$,

$$\langle P^{(n)}(i)a^i \rangle = \langle P^{(k+1)}(i)a^i \rangle = \langle (\alpha i^{k+1} + P^{(k)}(i))a^i \rangle = \langle \alpha i^{k+1}a^i + P^{(k)}(i)a^i \rangle$$

The equations hold because a polynomial of degree $(k + 1)$ is $\alpha i^{k+1}$ plus a polynomial of degree $k$. Owing to the inductive hypothesis, the second term $P^{(k)}(i)a^i$ is annihilated by $(E-a)^n = (E-a)^{k+1}$, therefore,

$$(E-a)^{k+2}\langle \alpha i^{k+1}a^i + P^{(k)}(i)a^i \rangle = (E-a)^{k+1}(E-a)\langle \alpha i^{k+1}a^i \rangle$$

If $(i+1)^{k+1}$ is expanded, it will be $i^{k+1} + (k+1)i^k + (k+1)^2i^{k-1} + \cdots = i^{k+1} + P^{(k)}(i)$ where $P^{(k)}(i)$ is a polynomial in $i$ of degree $k$. Therefore,

$$(E-a)^{k+1}\langle \alpha(i+1)^{k+1}a^{i+1} - \alpha i^{k+1}a^{i+1} \rangle = (E-a)^{k+1}\langle \alpha(i^{k+1} + P^{(k)}(i))a^{i+1} - \alpha i^{k+1}a^{i+1} \rangle$$

According to the inductive hypothesis, $\langle \alpha P^{(k)}(i)a^{i+1} \rangle$ can be annihilated by $(E-a)^{k+1}$, therefore

$$(E-a)^{k+1}\langle \alpha P^{(k)}(i)a^{i+1} \rangle = 0$$

Finally, combining all aforementioned equations, we can conclude that

$$(E-a)^{k+2}\langle P^{(k+1)}(i) \rangle = 0$$

This completes the proof by induction.
6. (20pts) Use the operator method and annihilators to prove Master Theorem is correct.

(a) When \( \frac{af(n)}{f(n)} > 1 \):

We let \( n = t_i, n/b = t_{i-1} \). Similar to the previous problem, \( t_i = b^i \), and we let

\[
F(i) = T(n) = T(b^i) = aT(b^{i-1}) + f(n) = aF(i-1) + f(n)
\]

For \( f(n) \), we have \( f(n) < af\left(\frac{n}{b}\right) \), therefore we let \( f(n) = \frac{a}{K} f\left(\frac{n}{b}\right) \) for some \( K > 1 \). Similar to above, we let \( G(i) = f(n) = \frac{a}{K} f\left(b^{i-1}\right) = \frac{a}{K} G(i-1) \), where \( n = b^i \) and \( i = \log_b n \). Then, \( G(i) = \left(\frac{a}{K}\right)^i \), and

\[
F(i) = aF(i-1) + \left(\frac{a}{K}\right)^i
\]

whose annihilator is \( (E - a)(E - \frac{a}{K}) \).

Since \( K > 1, a \neq \frac{a}{K} \), and the generic solution is \( F(i) = c_1 a^i + c_2 \left(\frac{a}{K}\right)^i \). Because \( K > 1, \frac{a}{K} < a \), and the growth rate of \( F(i) \) is dominated by \( c_1 a^i \), and \( F(i) = \Theta(a^i) \), which also means \( T(n) = \Theta(a \log_b n) = \Theta(n \log_b a) \).

(b) When \( \frac{af(n)}{f(n)} < 1 \):

Similarly, \( f(n) = af\left(\frac{n}{b}\right) \) for some \( K < 1 \), and \( F(i) \)'s growth rate is dominated by \( c_2 \left(\frac{a}{K}\right)^i \) in this case. Then, \( F(i) = \Theta\left(\left(\frac{a}{K}\right)^i\right) \), which is \( \Theta\left(G(i)\right) \). Then, \( T(n) = \Theta\left(f(n)\right) \).

(c) When \( \frac{af(n)}{f(n)} = 1 \):

This corresponds to the case where \( K = 1 \). \( F(i) \)'s annihilator is \( (E - a)(E - a) = (E - a)^2 \). Then, the generic solution is \( F(i) = (c_1 i + c_2 a^i) \in \Theta(i a^i) = \Theta(\log_b n f(n)) \), because \( f(n) = G(i) = \Theta(a^i) \).