1. (10pts) Sort the following integers by using the ‘merge sort’, and show its procedure step by step similar to Section 1.2 in Lecture note 5.

14, 1, 4, 8, 2, 9, 10, 11, 3, 7, 15, 5, 13, 6, 12
2. (20pts) Get the Big-Theta notations of the following recurrences by using Master Theorem.

- \( T(n) = 3T\left(\frac{n}{3}\right) + n \)

- \( T(n) = 2T\left(\frac{n}{4}\right) + n^2 \)

- \( T(n) = 4T\left(\frac{n}{2}\right) + \log n \)
3. (30pts) You are riding a motorcycle from Chicago to New York, and you found there are $n$ gas stations on the route. Because the tank of your motorcycle is small, you need to refuel the bike frequently, and you can pass at most 1 gas station without refuelling. Let’s see how many refuel strategies you can have throughout your trip to New York.

- Let $S(n)$ be the total number of refuel strategies. Identify the recurrence of $S(n)$.

- What is the annihilator of $S(n)$?

- What is the exact solution of $S(n)$?
4. (20pts) What is the general solution of

\[ a_n = 3a_{n-1} - 2a_{n-2} + F(n) \]

- if \( F(n) = 3^n \)?

- if \( F(n) = 2^n \)?
5. (20pts) Prove by induction on $n$ that the operator $(E-a)^{n+1}$ annihilates any sequence $\langle P^{(n)}(i)a^i \rangle$ where $P^{(n)}(i)$ is a polynomial in $i$ of degree $n$. 
6. (20pts) Use the operator method and annihilators to prove Master Theorem is correct.