Illinois Institute of Technology
Department of Computer Science

Solutions to Second Examination

CS 330 Discrete Structures
Fall Semester, 2009

1. **Mathematical Induction.** Prove by induction that $\left(\frac{n}{3}\right)^n < n! < \left(\frac{n}{2}\right)^n$ if $n \geq 6$. You may assume that $2 < \left(1 + \frac{1}{n}\right)^n < 3$ for all $n \geq 0$.

   Base case $n = 6$: $\left(\frac{6}{3}\right)^6 = 2^6 = 64$, $6! = 720$, and $\left(\frac{6}{2}\right)^6 = 3^6 = 729$.

   Induction: We need to prove that if $\left(\frac{n}{3}\right)^n < n! < \left(\frac{n}{2}\right)^n$ then $\left(\frac{n+1}{3}\right)^{n+1} < (n+1)! < \left(\frac{n+1}{2}\right)^{n+1}$.

   For the lower bound, since $\left(\frac{n}{3}\right)^n < n!$, we know that $(n+1)\left(\frac{n}{3}\right)^n < (n+1)!$, and we want to show that $\left(\frac{n+1}{3}\right)^{n+1} \leq (n+1)\left(\frac{n}{3}\right)^n$ when $n \geq 6$.

$$\begin{aligned}
\left(\frac{n+1}{3}\right)^{n+1} &\leq (n+1)\left(\frac{n}{3}\right)^n \\
\frac{1}{3}\left(\frac{n+1}{3}\right)^n &\leq \left(\frac{n}{3}\right)^n \\
\left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n &\leq 3
\end{aligned}$$

   For the upper bound, since $\left(\frac{n}{2}\right)^n > n!$, we know that $(n+1)\left(\frac{n}{2}\right)^n > (n+1)!$, and we want to show that $\left(\frac{n+1}{2}\right)^{n+1} \geq (n+1)\left(\frac{n}{2}\right)^n$ when $n \geq 6$.

$$\begin{aligned}
\left(\frac{n+1}{2}\right)^{n+1} &\geq (n+1)\left(\frac{n}{2}\right)^n \\
\frac{1}{2}\left(\frac{n+1}{2}\right)^n &\geq \left(\frac{n}{2}\right)^n \\
\left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n &\geq 2
\end{aligned}$$

2. **Growth rates.**

   (a) Is $n/H_n \in O(n)$? Prove your answer.
   Yes—we saw in class that $H_n \geq \ln n + 1/n$, so that $H_n \geq \ln n$. Therefore $n/H_n \leq n/\ln n = o(n) = O(n)$.

   (b) Is $\binom{2n}{n} \in O(2^n)$? Prove your answer.
   No. We can calculate $\binom{2n}{n}$ using Stirling's formula; ignoring lower order terms (which we never discussed):

$$\begin{aligned}
\binom{2n}{n} &= \frac{2n!}{(n!)^2} \\
&\approx \frac{c\sqrt{2n}(2n/e)^{2n}}{(c\sqrt{n}(n/e)^n)^2} \\
&= C4^n/\sqrt{n}
\end{aligned}$$

(for some constant C) which grows much faster than 2^n.

The above answer is more-or-less what I expected (though I did the calculation incorrectly and thought the answer was "Yes," not "No"!). It can be made precise using a fuller version of Stirling's formula, but I didn't expect that in an answer because we didn't talk about it.

The TA thought that we could use bounds such as those in the first exam problem, but we'd need sharper lower and upper bounds to make the argument work. Without a calculator, it would be hard to work that out.

One student began along the following lines: Expand $\binom{2n}{n}$ as

$$\begin{aligned}\binom{2n}{n} &= \frac{2n!}{(n!)^2} \\ &= \frac{1 \cdot 2 \cdot 3 \cdots 2n}{1^2 \cdot 2^2 \cdot 3^2 \cdot n^2} \\ &= \frac{(n+1)(n+2)(n+3)\cdots 2n}{1 \cdot 2 \cdot 3 \cdots n} \\ &= \frac{n+1}{1}\frac{n+2}{2}\frac{n+3}{3}\cdots\frac{2n}{n}\end{aligned}$$

It's a nice idea and can be continued to get a loose, but adequate, lower bound.

Another student had a clever idea, but I can't see how to make it work: Use the Vandermonde's identity (see the class notes)

$$\sum_{i=0}^{n}\binom{n}{i}^2 = \binom{2n}{n}.$$

I'll give 5 bugs to any student who works out a correct, complete proof along any of these three these lines of thought (give it to me in writing by next Wednesday).

All in all, this problem turned out to be trickier than I anticipated when I wrote it. Incidently, I was at a lecture on Thursday in which the speaker said $\binom{2n}{n} = \Theta(4^n)$; is that correct? Two bugs to any student with a correct answer and reasoning (again, in writing by next Wednesday).

3. **Rules of Sum Product.** Use the variation of the Rule of Product to prove that the number of combinations of n things taken k at a time is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

by taking event E as the number of permutations of n items.

This problem was mentioned as a "good exam question" in class on September 9 and 14 when we did it taking E as the number of permutations of n things *taken k at a time*. We viewed E as a compound event $E = E_1 \cap E_2$ in which E_2 is the event of choosing which k of n rooms will be occupied and E_1 is the event of arranging k guests in the occupied rooms.

Here we take event E_2 to be the formation of a combination of n items taken k at a time, so that $e_2 = \binom{n}{k}$ (that's the definition of that symbol). We can form a permutation of n items in $n!$ ways, so $e = n!$. A permutation of n items can be formed by choosing k of the positions (event E_2), and then (event E_1) filling them with a permutation of items $1, \ldots, k$ and filling the remaining $n-k$ positions with a permutation of items $k+1, \ldots, n$. Event E_1 can occur in $k!(n-k)!$ ways (by the Rule of Product). The variation of the Rule of Product then tells us that $e_2 = E/e_1 = \frac{n!}{k!(n-k)!}$.

4. **Binomial Coefficients.**

   Find the coefficient of x^{31} in each of the following polynomials. You need not simplify powers, factorials, or binomial coefficients.

   (a) $(1+2x)^{100}$
   By the binomial theorem, $2^{31}\binom{100}{31}$.

   (b) $(1-x)^{-5}$
   By the binomial theorem extended to negative exponents, $\binom{5+31-1}{31} = \binom{35}{31}$.

   (c) $(1-x^2+x^4-x^6+\cdots)^7$
   Zero, because the polynomial only has even powers of x.

   (d) $(1-5x)^{2n+1}$
   By the binomial theorem, $(-5)^{31}\binom{2n+1}{31} = -5^{31}\binom{2n+1}{31}$.

5. **Combinatorial Interpretation.**

   Give a combinatorial interpretation of the identity

$$\left[\sum_{k=0}^{n}\binom{n}{k}\right]^2 = \sum_{k=0}^{2n}\binom{2n}{k} = 2^{2n} = 4^n.$$

With n couples there are $2n$ people. By the Rule of Product, 2^{2n} is the number of subsets (of any size, including zero) of a set of the $2n$ people (each person can be in or out of the subset).

For each couple, we can have just the man, just the woman, both of them, or neither of them in the subset, four possibilities. There are n couples, so the Rule of Product tells us there are 4^n ways to select the subset of the couples.

Such a subset can be formed by choosing m, the number of men in the subset, then that number of men, followed by choosing w, the number of women in the subset, then that number of women. Using the Rules of Sum and Product,

$$\left[\sum_{m=0}^{n}\binom{n}{m}\right]\left[\sum_{w=0}^{n}\binom{n}{w}\right] = \left[\sum_{k=0}^{n}\binom{n}{k}\right]^2.$$

Such a subset can also be formed by choosing k, the number of people in the subset, followed by choosing that number of people from the $2n$ men and women. Using the Rule of Sum this is

$$\sum_{k=0}^{2n}\binom{2n}{k}.$$