Exam Statistics

64 students took the exam. The range of scores was 5–81, with a mean of 37.1, a median of 35, and a standard deviation of 18.7. Very roughly speaking, if I had to assign final grades on the basis of this exam only, above 59 would be an A (10), 40–59 a B (18), 20–39 a C (21), 10–19 a D (12), below 10 an E (3).

Problem Solutions

1. Mathematical Induction.

Base case: For $n = 2$, $\sum_{i=1}^{2} i^{-2} = 5/4 < 2 - 1/2$.

Inductive step: Assume that the inequality holds for $n \geq 2$ so that $\sum_{i=1}^{n} i^{-2} < 2 - 1/n$. We have $\sum_{i=1}^{n+1} i^{-2} = (n+1)^{-2} + \sum_{i=1}^{n} i^{-2} < (n+1)^{-2} + 2 - 1/n$ by induction. So we are done if $(n+1)^{-2} + 2 - 1/n < 2 - 1/(n+1)$, or $(n+1)^{-2} - 1/n < -1/(n+1)$. Multiplying both sides by $n(n+1)^2$ gives $n - (n+1)^2 < -n(n+1)$; that is, $-n^2 - n - 1 < -n^2 - n$, or $-1 < 0$, which is obviously true.

2. Growth rates.

(a) We calculate $\binom{2n}{4} = \frac{(2n)!}{4!(2n-4)!} = \frac{(2n)(2n-1)(2n-2)(2n-3)}{24}$ which grows proportionately to $n^4$ which is faster than $n^3$.

(b) We saw in class that $H_n = \ln n$ (plus lower order terms that we never discussed), so $n^{2H_n} = O(n^{2\ln n}) = O(ne^{\ln n}) = O(ne^{\ln n}) = O(n^{1+\ln 2})$. But $\ln 2 < 1$, so that $n^{2H_n} = O(n^2)$.


There are 16! placements of the cards in the 16 squares. Each card can be rotated $0^\circ$, $90^\circ$, $180^\circ$, or $270^\circ$, so there are $4^{16}$ possible orientations, for $4^{16} \times 16!$ placements. But each placement occurs 4 times because the entire $4 \times 4$ layout can be rotated $0^\circ$, $90^\circ$, $180^\circ$, or $270^\circ$. Hence there are $4^{15} \times 16!$ arrangements of the 16 cards.

4. Evaluation of Polynomials.

(a) We can calculate the polynomial by

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1: p ← 1
2: for i ← 1 to n do
3:   p ← 1 + p \times x
4: end for
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This clearly uses $n$ additions and $n$ multiplications.
(b) When \( x = 1 \), the value is \( n + 1 \). If division is permissible, and \( x \neq 1 \), we can write the expression as \( (x^{n+1} - 1)/(x - 1) \) and use \( \lceil \lg(n+1) \rceil \) multiplications to compute \( x^{n+1} \) with repeated squaring as follows:

\[
\begin{align*}
1: & \quad \text{power}(x, k) \\
2: & \quad \text{if } k = 0 \text{ then} \\
3: & \quad \text{return } 1 \\
4: & \quad \text{else if } k \mod 2 = 0 \text{ then} \\
5: & \quad \text{return } \text{power}(x, \lfloor k/2 \rfloor)^2 \\
6: & \quad \text{else} \\
7: & \quad \text{return } x \times \text{power}(x, \lfloor k/2 \rfloor)^2 \\
8: & \quad \text{end if} \\
9: & \quad \text{end power}
\end{align*}
\]

The total number of arithmetic operations to compute the given polynomial is then \( O(\log n) \).

5. Binomial Coefficients.

(a) From the Binomial Theorem, \( \binom{15}{10} \).

(b) Again from the Binomial Theorem, \( (-3)^{10} \binom{k}{10} = 3^{10} \binom{k}{10} \).

(c) Zero, because there is no \( x^{10} \) term—all the exponents are multiples of 3.

(d) The Binomial Theorem applied to negative exponents (page 6 of the notes from February 6–11) gives \( (-5)^{10} \binom{-n}{10} = 5^{10} \binom{n+9}{10} \).