Exam Statistics

80 students took the exam; 2 students did not. The range of scores was 5–95, with a mean of 50.16, a median of 48, and a standard deviation of 18.09. Very roughly speaking, if I had to assign final grades on the basis of this exam only, above 70 would be an A (13), 50–70 a B (28), 40–49 a C (22), 30–39 a D (9), below 30 an E (10).

Problem Solutions

1. Mathematical Induction.

(a) Basis step: For $k = 1$, $S_k$ becomes the statement “On the first call to recursive-gcd we have $u = r_0$ and $v = r_1$.” This is true by lines 2–3 in iterative-gcd.

Inductive step: Assume $S_k$ is true, show $S_{k+1}$ is true. Because $S_k$ is true, on the $k$th call to recursive-gcd we have $u = r_{k-1}$ and $v = r_k$. On the $k$th call, if $v = 0$, there is no $k + 1$st call; to have a $k + 1$st call we must have $v \neq 0$, so the else clause (statement 5 in recursive-gcd) is executed (to give the $k + 1$st call) with first parameter $v = r_k$ and second parameter $u \mod v = r_{k-1} \mod r_k = r_{k+1}$ by definition. That is, upon entering recursive-gcd for the $k$th time, the first parameter is $u = r_k$ and the second is $v = r_{k+1}$, so that $S_{k+1}$ holds.

(b) After the last iteration of the while loop in iterative-gcd, when $r_k = 0$, we return $r_{k-1}$. In recursive-gcd we return $u$ when $v = 0$; but $S_k$ tells us that $v = r_k$, so the value returned, $u$, must be $r_{k-1}$.

2. Growth rates.

(a) We calculate $\binom{2n}{n}$ using Stirling’s formula; ignoring lower order terms (which we never discussed):

$$\binom{2n}{n} = \frac{2n!}{(n!)^2} = \Theta \left( \frac{c\sqrt{2n(2n/e)^{2n}}}{(e\sqrt{n/n/e})^{2n}} \right) = \Theta \left( \frac{4^n}{\sqrt{n}} \right) = o(n^n).$$

(b) We saw in class (August 23) that $H_n = O(\log n)$, so $10n\sqrt{H_n} = O(n\sqrt{\log n})$ which is $\Theta(n \log_2 n)$.


This is an extraordinarily easy problem!

(a) The question is, in effect, for what value of $n$ (the number of rooms) is $n^2 = 9$ (the first clause) or $n(n - 1) = 6$ (the second clause). The answer is, of course, $n = 3$ rooms.

(b) Now the question is either for what value of $n$ is $n(n - 1) = 36$ (if the first clause is meant) or for what value of $n$ is $n^2 = 36$ (if the the second clause is meant). Since the latter case has no integer solution, the statement must be referring to the first clause, in which case the ideal apartment would have $n = 6$ rooms.
   If the recursion is expanded out, the computation is basically $1+1+...+1$, so there are $\binom{n}{k} - 1$ additions. This follows from the identity used, together with an induction on $n$ and $i$ that states “Computing $\binom{n}{i}$ by the above code takes $\binom{n}{i} - 1$ addition operations.”

5. Binomial Coefficients.
   (a) Zero, because there is no $x^{10}$ term.
   (b) From the Binomial Theorem, $2^{10}\binom{k}{10}$.
   (c) $(1 - x^{2} + x^{4} - x^{6} + \cdots) = (1 - x^{2})^{-1}$. Because $(x^{2})^{5} = x^{10}$, we use equation (12) on page 6 of the notes from September 13–18, with $n = -2$ and $k = 5$; the answer is $\binom{7}{5} = 21$.
   (d) From the Binomial Theorem applied to negative exponents (page 6 of the notes from September 13–18), $5^{10}\binom{2n+11}{12}$. 