56 students took the exam; there was one unexcused no-show (recorded as 0) and one excused student. The range of scores (excluding the no-show and the excused student) was 16–96, with a mean of 53.7, a median of 54, and a standard deviation of 19.3. Very roughly speaking, if I had to assign final grades on the basis of this exam only, above 70 would be an A (14), 50–69 a B (17), 35–49 a C (15), 23–34 a D (7), below 23 an E (3).

Problem Solutions

1. (a) There are \(k + 1\) places to insert the \(n - k\) empty spaces (\(k - 1\) between adjacent cars and one at each end). To prevent an SUV parking, at most one empty space can be inserted in these \(k + 1\) places. Hence the answer is \(\binom{k + 1}{n - k}\).

   (b) There are \(\binom{n}{k}\) ways that the \(k\) cars can park; because the cars park at random, all are equally probable. From part (a), there are \(\binom{k + 1}{n - k}\) ways for the \(k\) cars to park so that the SUV cannot park. Hence the probability that the SUV can park is \(1 - \frac{\binom{k + 1}{n - k}}{\binom{n}{k}}\).

2. (a) The probability that \(P\) kills \(Q\) on the first shot is \(p\). If \(P\) misses, probability \((1 - p)\), and then \(Q\) misses, probability \((1 - q)\), the probability that \(P\) kills \(Q\) on the second shot is \(p\), so the probability that \(P\) kills \(Q\) on the second shot is \((1 - p)(1 - q)p\). In general, the probability that \(P\) kills \(Q\) at the \(k\)th shot is \(p((1 - p)(1 - q))^k\), so the probability that \(P\) wins the duel is

\[
\sum_{k=0}^{\infty} p((1 - p)(1 - q))^k = p \sum_{k=0}^{\infty} [(1 - p)(1 - q)]^k = \frac{p}{1 - (1 - p)(1 - q)} = \frac{p}{p + q - pq},
\]

using the formula for the sum of a geometric progression (Theorem 1 on page 174 in Rosen).

One can also analyze it differently: Let \(f(p, q)\) be the probability that the first player, \(P\), wins. Then \(f(p, q) = p + (1 - p)(1 - f(q, p))\) so that by iterating this recurrence we get \(f(p, q) = p + (1 - p)(1 - (q + (1 - q)(1 - f(p, q)))) = p + (1 - p)(1 - q)f(p, q)\) and hence \(f(p, q) = p/(p + q - pq)\).

(b) The probability of the duel ending with \(P\) firing a single shot is \(p + (1 - p)q\) (either \(P\) kills \(Q\) on the first shot, or \(P\) misses and is killed by \(Q\) on \(Q\)’s first shot). In general, for \(P\) to fire \(k\) times, \(P\) must miss the first \(k - 1\) shots, as must \(Q\), and then kill \(Q\) on the \(k\)th shot; this has probability \((1 - p)^{k-1}(1 - q)^{k-1}p\). Or, \(P\) can miss the first \(k\) shots and be killed by \(Q\) on \(Q\)’s \(k\)th shot; this has probability \((1 - p)^k(1 - q)^{k-1}q\) so the expected number of shots fired by \(P\) is
\[
\sum_{k=1}^{\infty} k[(1-p)^{k-1}(1-q)^{k-1}p + (1-p)^k(1-q)^k] = p \sum_{k=0}^{\infty} (k+1)(1-p)^k(1-q)^k + (1-p)q \sum_{k=0}^{\infty} (k+1)(1-p)^k(1-q)^k
\]

\[
= (p + q - pq) \sum_{k=0}^{\infty} (k+1)(1-p)^k(1-q)^k = \frac{1}{p + q - pq},
\]

where the summation is evaluated using the method of Example 25 on pages 176–177 of Rosen.

3. Use Bayes’ Theorem,

\[
\Pr\{F|E\} = \frac{\Pr\{F\}\Pr\{E|F\}}{\Pr\{E\}},
\]

where \(E\) is the event of failing the second exam and \(F\) is the event of attending class regularly and doing all your own homework. Thus \(\Pr\{E\} = 1/2, \Pr\{E|F\} = 1/3, \text{ and } \Pr\{F\} = 4/5\). Therefore the probability that a student failed the exam but both attended class and did all his own homework is \((4/5)(1/3)/(1/2) = 8/15\).

4. | Annihilator | Growth Rate | Sample Recurrence |
<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>((E + 2)(E - 4))</td>
<td>(\Theta(4^n))</td>
<td>(S_n = 2S_{n-1} + 8S_{n-2})</td>
</tr>
<tr>
<td>((E - 5))</td>
<td>(\Theta(5^n))</td>
<td>(S_n = 5S_{n-1})</td>
</tr>
<tr>
<td>((E^2 - 1)(E - 1)^4 = (E - 1)^4(E + 1))</td>
<td>(\Theta(n^4))</td>
<td>(S_n = S_{n-2} + n^4)</td>
</tr>
<tr>
<td>((E - 2)^2)</td>
<td>(\Theta(n2^n))</td>
<td>(S_n = 2S_{n-1} + 2^n)</td>
</tr>
<tr>
<td>((E - 1/3))</td>
<td>(\Theta(3^{-n}))</td>
<td>(S_n = S_{n-1}/3)</td>
</tr>
<tr>
<td>((E - \phi)(E - \hat{\phi})(E - 1)^2)</td>
<td>(\Theta(\phi^n))</td>
<td>(S_n = S_{n-1} + S_{n-2} + n)</td>
</tr>
</tbody>
</table>

5. (a) The recurrence for the TA’s “improvement” is solved by the second case of the Master Theorem with \(a = 10, b = 4, \text{ and } f(n) = \Theta(n)\), \(T(n) = n^{\log_4 10}\). But \(\log_4 10 = 0.5 \log 10 = \lg \sqrt{10} > \lg 3\), so this “improvement” is inferior to the algorithm as presented in class.

(b) The analysis is the same for the new version of the TA’s algorithm, except \(a = 25\) and \(b = 8\); note that \(f(n)\) is still \(\Theta(n)\), albeit with a higher constant multiple. Thus by the second case of the Master Theorem \(T(n) = n^{\log_8 25}\). Furthermore, \(\log_8 25 = (1/3) \log 25 = \lg 25^{1/3} < \lg 27^{1/3} = \lg 3\), so this algorithm is an improvement asymptotically and Reingold was impressed.