Exam Statistics

69 students took the exam. The range of scores was 22–87, with a mean of 52.46, a median of 49, and a standard deviation of 16.95. Very roughly speaking, if I had to assign final grades on the basis of this exam only, above 69 would be an A (13), 50–69 a B (21), 40–49 a C (20), 29–39 a D (11), below 29 an E (4).

Problem Solutions

1. (a) The \( n + 1 \) vertices can be in any order in the main list of the adjacency structure—that is, \((n+1)!\) orders. The individual vertex lists of edges are as follows: The \( n \) edges connected to \( v_0 \) can be in \( n! \) orders. The two edges connected to \( v_1 \) and to \( v_n \) can each be in 2 orders. The 3 edges connected to each of the \( n - 2 \) vertices \( v_2, \ldots, v_{n-1} \) can be in \( 3! = 6 \) orders. Hence the total number of adjacency structures is \((n+1)! \times n! \times 2 \times 2 \times 6^{n-2}\).

(b) Here are the \( d \) values (in red):

\[
\begin{align*}
0_{v_1} & & v_2 & & 1 \\
v_0 & & v_1 & & v_2 \\
v_0 & & v_1 & & v_3 \\
v_0 & & v_1 & & v_n \\
v_0 & & v_1 & & v_{n-1} \\
v_0 & & v_1 & & v_{n-2} \\
v_0 & & v_1 & & \vdots \\
2_{v_n} & & 2 \\
2_{v_{n-2}} & & 2 \\
2_{v_{n-1}} & & 2
\end{align*}
\]
2. Delete lines numbered 4, 6, 16, 19, and 24 (all grayed out); then add the two lines in red:

1: function DFS(G)
2:   for all \( u \in V[G] \) do
3:     color\([u]\) ← WHITE
4:     π\([u]\) ← NIL
5:   end for
6:   time ← 0 components ← 0
7:   for all \( u \in V[G] \) do
8:     if color\([u]\) = WHITE then
9:         components ← components + 1
10:        DFS-visit(u)
11:   end if
12: end function

14: function DFS-visit(u)
15:     color\([u]\) ← GRAY
16:     d\([u]\) ← time ← time + 1
17:     for all \( v \in Adj[u] \) do
18:       if color\([v]\) = WHITE then
19:         π\([v]\) ← u
20:        DFS-visit(v)
21:   end if
22: end for
23:     color\([u]\) ← BLACK
24:     f\([u]\) ← time ← time + 1
25: end function

3. The modified Euler’s formula for a planar graph consisting of \( |V| \) vertices, \( |E| \) edges, \( |F| \) faces, and \( k \) connected components is

\[
|V| - |E| + |F| = k + 1. \quad (1)
\]

The inductive proof of correctness is as follows. For \( k = 1 \), this is just Euler’s formula as given in class on November 8.

Suppose equation (1) is true for \( k \) components; we must prove it then also holds for \( k + 1 \) components. Consider such a graph \( G \) of \( |V| \) vertices, \( |E| \) edges, \( |F| \) faces, and \( k + 1 \) connected components. Pick any connected component \( C \) in \( G \); suppose \( C \) taken as a separate graph has \( |V_C| \) vertices, \( |E_C| \) edges, and \( |F_C| \) faces. Because \( C \) is a connected graph (that is, consists of a single component),

\[
|V_C| - |E_C| + |F_C| = 2, \quad (2)
\]

by Euler’s formula. Note that

\[
|F| = |F - F_C| + |F_C|.
\]
By induction, formula (1) works for $G - C$ (that is, $G$ with component $C$ deleted), so

$$|V - V_C| - |E - E_C| + |F - F_C| + 1 = k + 1,$$

where we must add one to the lefthand side because $|F - F_C|$ omits the infinite face which is in both $F$ and $F_C$.

Adding (2) and (3) gives us

$$|V| - |E| + |F| = k + 2,$$

as needed for $k + 1$ components.

4. (a) This set is regular. We can recognize strings with a multiple of three zeros with the 3-state FSM

![3-state FSM diagram]

and recognize strings with an odd number of ones with the 2-state FSM

![2-state FSM diagram]

We can then take the intersection of languages recognized by these two FSMs, a 6-state FSM.

(b) The set of English words used in the examination is a finite set; all finite sets are regular.
(c) The proof is by contradiction. Suppose $L$ were regular. Following the hint, we note that because regularity is preserved by complementation, $L$ is regular if and only if $\Sigma^* - L$ is regular. Because $a^*b^*$ is a regular expression, the set of strings that it describes is regular. Finally, the intersection of two regular sets is regular, so $L$ is regular if and only if $(\Sigma^* - L) \cap a^*b^*$ is regular. But $(\Sigma^* - L) \cap a^*b^* = \{a^n b^m | n = m\}$ which is not regular, as we saw in class on November 20, by the pigeonhole principle ("pumping").

5. (a) All strings of zeros and ones that contain an odd number of zeros; compare this with the solution to the latter part of 4(a) above!
(b) $1^*0(1*01*01^*)^*$
(c) All strings of zeros and ones, including the empty string; that is, $(0 + 1)^*$. 