Print your name and student ID, neatly in the space provided below; print your name at the upper right corner of every page. Please print legibly.

Name:
Student ID:

This is an open book exam. You are permitted to use the textbook (hardcopy only), any class handouts, anything posted on the web page, any of your own assignments, and anything in your own handwriting. Foreign students may use a dictionary. Nothing else is permitted: No calculators, laptops, cell phones, Internet-enabled watches, Ipads, Ipods, communicators, GPSes, etc.!

Do all five problems in this booklet. All problems are equally weighted, so do not spend too much time on any one question.

Show your work! You will not get partial credit if the grader cannot figure out how you arrived at your answer.

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1. **Mathematical Induction.** In class on August 28 we looked at two versions of Euclid’s algorithm to compute the greatest common divisor of two positive integers \(u\) and \(v\); the iterative and recursive computations were, respectively:

```
1: function iterative-gcd(u, v)
2:     r_0 \leftarrow u
3:     r_1 \leftarrow v
4:     k \leftarrow 1
5:     while r_k > 0 do
6:         k \leftarrow k + 1
7:         r_k \leftarrow r_{k-2} \mod r_{k-1}
8:     end while
9:     return r_{k-1}
10: end function
```

```
1: function recursive-gcd(u, v)
2: if v = 0 then
3:     return u
4: else
5:     return recursive-gcd(v, u \mod v)
6: end if
7: end function
```

Let \(S_k\) be the statement

On the \(k\)th call to \textit{recursive-gcd}, \(u = r_{k-1}\) and \(v = r_k\).

The originating—that is, external—call to \textit{recursive-gcd} is the first call \((k = 1)\).

(a) Prove by induction on \(k\) that \(S_k\) holds for all integer \(k > 0\).

(b) Use the truth of \(S_k\) to prove the two functions return the same value.
2. Growth rates.

(a) Does \( \binom{2n}{n} \) grow slower than, the same as, or faster than \( n^n \)? (At the bottom of page 3 in the notes of the September 6 lecture, a version of this is given as an exercise.)

(b) Is \( 10n\sqrt{T_n} = O(n \log_2 n^2) \)?

In both cases, prove your answer.
3. **Rules of Sum and Product.**

In the 2014 novel *The Rosie Effect* by Graeme Simsion, a paragraph on page 2 reads

> In our apartment, there were nine possible combinations of locations for two people, of which six involved [the two of] us being in different rooms. In our ideal apartment...there would have been thirty-six possible combinations...

(a) How many rooms are in their apartment?
(b) How many rooms are in their ideal apartment?

*(Hint: What does “thirty-six possible combinations” tell you is being asked?)*

In both parts, explain/prove your answer!

Consider the following computation of \( \binom{n}{i} \):

1: function \( \text{Combination}(n, i) \)
2: if \( i = 0 \) or \( i = n \) then
3: return 1
4: else
5: return \( \text{Combination}(n - 1, i - 1) + \text{Combination}(n - 1, i) \)
6: end if
7: end function

Analyze the number of additions (the + operations in line 5) needed to compute \( \binom{n}{i} \).
5. **Binomial Coefficients.**

Derive the coefficient of $x^{10}$ in each of the following polynomials. You need not simplify powers, factorials, or binomial coefficients.

(a) $(1 - x)^9$
(b) $(1 + 2x)^k$
(c) $(1 - x^2 + x^4 - x^6 + \cdots)^2$
(d) $(1 - 5x)^{-2n}$