Print your name and student ID, neatly in the space provided below; print your name at the upper right corner of every page. Please print legibly.

Name:
Student ID:

This is an open book exam. You are permitted to use the textbook (hardcopy only), any class handouts, anything posted on the web page, any of your own assignments, and anything in your own handwriting. Foreign students may use a dictionary. Nothing else is permitted: No calculators, laptops, cell phones, Internet-enabled watches, Ipads, Ipods, communicators, GPSes, etc.!

Do all four problems in this booklet. All problems are equally weighted, so do not spend too much time on any one question.

Show your work! You will not get partial credit if the grader cannot figure out how you arrived at your answer.
1. Rolling a Die

(a) In the 2013 book *Math on Trial: How Numbers Get Used and Abused in the Courtroom* by Leila Schneps and Coralie Colmez, a discussion on page 190 gives the probability of throwing exactly 3 sixes in 6 rolls of a die as 20/216. This is wrong! Derive the correct value.

(b) Derive the probability of throwing at least 3 sixes in 6 rolls of a die.

You do not need to simplify expressions or compute approximations.
2. Double Rotations

Consider the double rotation transformation of a binary search tree (that is, rotating clockwise around C, then counter-clockwise around A):

(a) Explain why $\alpha_A = \beta_B \beta_A$.
(b) Express $\alpha_C$ in terms of $\beta_A$, $\beta_B$, and $\beta_C$.

(Hint: This was suggested as an exercise at the end of the lecture notes from October 2.)
3. **Linear Recurrences**

Fill in the ten missing entries in the following table:

<table>
<thead>
<tr>
<th>Annihilator</th>
<th>Growth Rate</th>
<th>Sample Recurrence</th>
</tr>
</thead>
<tbody>
<tr>
<td>((E + 1)(E - 2))</td>
<td>(\Theta(3^n))</td>
<td>(S_n = 3S_{n-1})</td>
</tr>
<tr>
<td>((E - 5)^2)</td>
<td>(\Theta(n^3))</td>
<td>(S_n = S_{n-2} + 2^n)</td>
</tr>
<tr>
<td>((E - \varphi)(E - \hat{\varphi})(E - 1))</td>
<td>(\Theta(n^3))</td>
<td>(S_n = S_{n-1} + S_{n-2} + 2)</td>
</tr>
</tbody>
</table>

Recall that \(\varphi\) (the Golden Ratio) and \(\hat{\varphi}\) are \(\frac{1 \pm \sqrt{5}}{2}\), respectively.
4. **Divide-and-Conquer**

To demonstrate recursion and divide-and-conquer, Professor Reingold decided to write a program to compute $x^n$.

(a) His first attempt was

```
function Power(x, n)
    1: if n = 0 then
    2:     return 1
    3: else if n is odd then
    4:     return x * Power(x, ⌊n/2⌋) * Power(x, ⌊n/2⌋)
    5: else
    6:     return Power(x, ⌊n/2⌋) * Power(x, ⌊n/2⌋)
    7: end if
end if
```

Analyze the time required by this algorithm.

(b) His second attempt was

```
function Power(x, n)
    1: if n = 0 then
    2:     return 1
    3: else
    4:     integer t ← Power(x, ⌊n/2⌋)
    5: if n is odd then
    6:     return x * t * t
    7: else
    8:     return t * t
    9: end if
10: end if
```

Analyze the time required by this algorithm.
4. Divide-and-Conquer, continued.