4.1 Recurrence Relations - Key Concepts

- Sequences of the form $a_0, a_1, \ldots$ can often be defined recursively, using a recurrence relation having
  - One or more base cases: Exact values for one or more initial sequence terms, and,
  - A recurrence expression $a_n$ as a recursive formula (one involving previous sequence terms).

A recurrence relation produces a sequence. A sequence can often be easily reformulated as a recurrence relation.

Example (Factorial Sequence)

- Define the sequence $\{n!\}_{n=0}^{\infty}$ recursively: Base case: $0! = 1$, recursive case: $n! = (n-1)! \cdot n$, for $n \geq 1$.

- Expressed as the sequence $\{a_n\}_{n=0}^{\infty}$, we have $a_0 = 1$ and $a_n = a_{n-1} \cdot n$, for $n \geq 1$. Using bottom-up iteration to compute the sequence of values, we get

<table>
<thead>
<tr>
<th>index $n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>formula for $a_n$</td>
<td>base case</td>
<td>$a_0 \cdot 1$</td>
<td>$a_1 \cdot 2$</td>
<td>$a_2 \cdot 3$</td>
<td>$a_3 \cdot 4$</td>
<td>$a_4 \cdot 5$</td>
<td>$a_5 \cdot 6$</td>
</tr>
<tr>
<td>value of $a_n$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>6</td>
<td>24</td>
<td>120</td>
<td>720</td>
</tr>
</tbody>
</table>

Example (Arithmetic Sequence)

- The arithmetic sequence $\{a_n\}_{n=0}^{\infty}$ with first term $a$ and common difference $d$ is defined as follows: base case $a_0 = a$, recursive case $a_n = a_{n-1} + d$. If $a = 23$ and $d = 3$, bottom-up iteration gives us

<table>
<thead>
<tr>
<th>index $n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>formula for $a_n$</td>
<td>base case</td>
<td>$a_0 + 3$</td>
<td>$a_1 + 3$</td>
<td>$a_2 + 3$</td>
<td>$a_3 + 3$</td>
<td>$a_4 + 3$</td>
<td>$a_5 + 3$</td>
</tr>
<tr>
<td>value of $a_n$</td>
<td>23</td>
<td>26</td>
<td>29</td>
<td>32</td>
<td>35</td>
<td>38</td>
<td>41</td>
</tr>
</tbody>
</table>

Questions

1. Complete this definition of a general geometric sequence with first term $a$ and common ratio $r$: Base case: $a_0 = a$; recursive case: $a_n = \ldots$. Generate the first six terms of a geometric sequence with $a = 3$ and common ratio $r = -2$.

2a. Generate the first six terms for the sequence $\{a_n\}_{n=3}^{\infty}$ with definition $a_3 = 0$ and $a_n = a_{n-1} + 2n - 7$.

2b. Guess a (non-recursive) formula for $a_n$ and verify that it satisfies the recurrence.

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3a. Draw all the legal tilings for a 2×3 rectangle. For this problem, we want to tile a rectangular grid using three possible tile shapes: a 2×1 rectangle, and no tiles can overlap. Below are drawings of the four basic shapes plus the two possible 2×1 tilings.  

Let \( t_n \) be the number of tilings for a 2×\( n \) rectangle. By inspection, there are 2 tilings for a 2×1 rectangle, and so \( t_1 = 2 \).

### Question

3. For this problem, we want to tile a rectangular grid using three possible tile shapes: a 2×2 square, a 2×1 rectangle (if used vertically; it can also be used as a horizontal 1×2 rectangle), and a 1×1 square. (A tiling must cover every square of the rectangle, and no tiles can overlap.) Below are drawings of the four basic shapes plus the two possible 2×1 tilings. In addition, two of the possible 3×2 tilings are disallowed and are also shown below.

<table>
<thead>
<tr>
<th>Four Basic Shapes</th>
<th>The Two 2×1 Tilings</th>
<th>The Disallowed 3×2 Tilings</th>
</tr>
</thead>
<tbody>
<tr>
<td>S S V V H H s s</td>
<td>V₁ s₁ H₁ H₁ s₁ s₁</td>
<td>H₁ H₁ H₁ s₁ s₁ s₁</td>
</tr>
<tr>
<td>S S V H s</td>
<td>V₁ s₁ H₁ H₁ s₁ s₁</td>
<td>H₁ H₁ H₁ s₁ s₁ s₁</td>
</tr>
</tbody>
</table>

Let \( t_n \) be the number of tilings for a 2×\( n \) rectangle. By inspection, there are 2 tilings for a 2×1 rectangle, and so \( t_1 = 2 \).

3a. Draw all the legal tilings for a 2×2 rectangle to determine \( t_2 = \) _________ .

3b. For \( n \geq 3 \), one way to get a 2×\( n \) tiling is to take each of the \( t_{n-1} \) tilings for 2×(\( n-1 \)) and adding one column. There are _________ ways to add the last column, so \( t_n \geq t_{n-1} \times \) _________ .
3c. For \( n \geq 3 \), another way to get a \( 2 \times n \) tiling is to take each of the \( t_{n-2} \) tilings for a \( 2 \times (n-2) \) rectangle and adding two columns. There are \( \square \) ways are there to add those two columns, but \( \square \) of those ways are covered by adding 1 column and then 1 more column, so \( t_n \geq t_{n-2} \times \square \).

3d. For \( n \geq 3 \), how else can we get a \( 2 \times n \) tiling from smaller tilings? Are these cases subsumed by the cases we already have for extending \( 2 \times (n-1) \) and \( 2 \times (n-2) \) tilings?

3e. Using all this information, give a definition for the sequence \( t_n \).

4.2 Mathematical Induction

- Mathematical Induction is a powerful, rigorous technique for proving that a predicate \( P(n) \) is true for every natural number \( n \), no matter how large. It’s often characterized using a ladder-climbing or domino-effect analogy. The first form of induction is as follows:

The First Principle of Mathematical Induction

\[
P(0) \\
(\forall n \geq 0) (P(n) \rightarrow P(n+1))
\]

\[\therefore (\forall n \geq 0) P(n)\]

Outline of an Inductive Proof

- Say we want to prove \( \forall n \in \mathbb{N} \ P(n) \):
  - **Base case (or basis step):** Prove \( P(0) \).
  - **Inductive step:** Prove \( (\forall n)(P(n)\rightarrow P(n+1)) \)
    - Using a direct proof: Let \( n \in \mathbb{N} \), assume the inductive hypothesis, \( P(n) \). Under this assumption, prove \( P(n+1) \).
    - Or using an indirect proof (contraposition or contradiction)
  - Inductive inference rule then gives \( \forall n \in \mathbb{N} \ P(n) \).

Questions

4. Prove that the sum of the first \( n \) odd positive integers is \( n^2 \). That is, prove: \( (\forall n \geq 1) \sum_{i=1}^{n} (2i - 1) = n^2 \).
5. Complete the following proof of the statement For all positive integers \( n \geq 4 \), we find \( 2n + 3 \leq 2^n \), using mathematical induction. Lines with * have something missing.

Define the predicate \( P(n) \) to be \( 2n + 3 \leq 2^n \) with domain \( n \in \mathbb{N} = \{0, 1, 2, \ldots\} \)

* Basis Step: \( P(\quad) \) is the statement:
  * Which simplifies to \( \quad \), so \( P(\quad) \) is true.

* Inductive Step: Let \( k \) be an integer with \( k \geq 1 \)
  * Assume \( P(k) \) is true. By definition, this means
    * Because \( k \geq 1 \), we have \( 2 \leq 2^k \).
  * Add \( 2 \leq 2^k \) to the inequality \( P(k) \) to obtain
  * Re-express both sides to get
    * Which is the inequality for \( P(k+1) \), so \( P(k+1) \) is true
  * By mathematical induction then,

6. Prove that \( \forall n > 0, n < 2^n \)