CS330 Activities for Lecture 11

5.1 Sum and Product Rules

- Let $m$ be the number of ways to do task 1 and $n$ the number of ways to do task 2 (with each number independent of how the other task is done), and assume that no way to do task 1 simultaneously also accomplishes task 2.
- **The sum rule**: The task “Do either task 1 or task 2, but not both” can be done in $m + n$ ways.
- **The product rule**: The task “Do both task 1 and task 2” can be done in $m \times n$ ways.
- Combining the Product and Sum Rules of Counting allows a broad number of counting problems to be solved.

Questions

1. A sandwich shop has four kinds of fillings (ham, turkey, cheese, and avocado) and three kinds of bread (white, whole wheat, and rye). How many distinct sandwiches are possible?

2. In Internet Protocol v.4, an IP Address has a header, netid and hostid.
   - **Class A**: Header 0, 7-bit netid (which can’t be all 1’s), 24-bit hostid
   - **Class B**: Header 10, 14-bit netid, 16-bit hostid.
   - **Class C**: Header 110, 21-bit netid, 8-bit hostid.
   - A hostid can’t be all 0s or all 1s
   How many valid IP addresses are there?

3. Back to the sandwich shop. They’re offering a special today: Your sandwich comes with either soup or salad. There’s chicken noodle soup and tomato soup, and there are caesar, house, and garden salads. How many different lunch orders are there?

5.2 Inclusion-Exclusion Principle

- The sum rule says that the $|A \cup B|$ (the size of $A$ union $B$) = $|A| + |B|$ if $A$ and $B$ are disjoint ($A \cap B = \emptyset$).
- If $A$ and $B$ are of finite size but not disjoint, then $|A| + |B| > |A \cup B|$ because it double-counts the overlap $A \cap B$. To compensate, we have to subtract the size of $A \cap B$ from $|A| + |B|$ to get $|A \cup B|$; this is the **inclusion-exclusion principle** (a.k.a. subtraction rule, fixing over-counting).
- Inclusion-Exclusion Principle: If sets $A$ and $B$ are both finite, then $|A \cup B| = |A| + |B| - |A \cap B|$.
- The zybook includes a nice animation; here’s a text-oriented argument:
  - $A = (A - B) \cup (A \cap B)$ so $|A| = |A - B| + |A \cap B|$, since there’s no overlap
  - Similarly $B = (B - A) \cup (A \cap B)$, and $|B| = |B - A| + |A \cap B|$.
  - $A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$

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So $|A \cup B| = |A - B| + |B - A| + |A \cap B|$ (since these don’t overlap)

So $|A \cup B| + |A \cap B| = |A - B| + |A \cap B| + |B - A| + |A \cap B| = |A| + |B|

So $|A \cup B| = |A| + |B| - |A \cap B|$

Questions
4. Hypothetical rules for passwords:
   - Passwords must be 2 characters long.
   - Each password must be a letter a-z, a digit 0-9, or one of the 10 punctuation characters !@#$%^&*()
   - Each password must contain at least 1 digit or punctuation character.
   - A legal password has a digit or punctuation character in position 1 or position 2.

How many legal passwords are there? Since the cases overlap, the inclusion-exclusion principle applies.

5. (Palindromic bit strings)
5a. How many binary strings of length 6 are palindromes?

5b. How many binary strings of length 6 begin with two 0s?

5c. How many binary strings of length 6 either are a palindrome or begin with two 0s?

6. A five-card hand is dealt from a standard playing deck (each card has one of four suits (spades, hearts, diamonds, clubs) and one of thirteen face values (2, 3, …, 10, Jack, Queen, King, Ace).
6a. How many hands have at least two 7's?

6b. How many hands have exactly two 7's or exactly two 8's?
5.3 Pigeonhole Principle

- When there are more pigeons than pigeonholes, some pigeonhole is forced to contain at least 2 pigeons. The more pigeons there are, the higher the number of pigeons you will be able to find in one of the pigeonholes.
- The basic principle seems simple, but the proof is tricky, and subtle applications of the principle can be powerful.
- **The Pigeonhole Principle.** If a function $f$ has a domain of size at least $n+1$, for positive integer $n$, and a target of size at most $n$, then there exist two elements in the domain that map to the same element in the target (i.e., the function is not one-to-one).
- **The generalized pigeonhole principle.** Consider a function whose domain has $n$ elements and whose target has $k$ elements, for $n$ and $k$ positive integers. Then there is an element $y$ in the target such that $f$ maps at least $\lceil n/k \rceil$ elements in the domain to $y$.

**Questions**

7. How many classes must be scheduled in 4 time slots to guarantee a class conflict?

8. A chessboard is an $8 \times 8$ grid with 64 squares. A rook is a chess piece that can move any number of spaces left, right, up, or down. Call a square “covered” if a rook can move onto it. Two rooks conflict if one is sits on a square covered by another rook. (I.e., they conflict if one rook can move onto the other rook.)

8a. Place a rook onto an empty board. How many squares does the rook cover?

8b. Now place a second rook down so that the two rooks don’t conflict. How many squares are covered by the two rooks?

8c. Try generalizing this: If $n$ rooks are placed so that none conflict with each other, how many squares are covered by them?
8d. How many rooks can you place on a chessboard without forcing a conflict between two rooks?

8e. What happens when one more rook is placed than the number you determined in the previous part?

9. If there are 27 students taking 4 versions of an exam,
9a. Must there be 6 students who take the same version? If not, give a counterexample.

9b. Must there be 7 students who take the same version? If not, give a counterexample.

9c. Must there be 8 students who take the same version? If not, give a counterexample.