5.6 Counting Subsets

- In a permutation, we have a sequence of elements in an order. At times, we don’t care about the order, just the members of the set of values.
- **Definition** The number of ways to choose \( r \) items from a set of \( n \) items without ordering them is defined to be \( C(n, r) \), read “\( n \) choose \( r \)”. We also write \( \binom{n}{r} \) for \( C(n, r) \).
- Since a combination is an unordered collection of values and a permutation is an ordered sequence of values, we can build an \( r \)-permutation by (1) choosing \( r \) elements from our set of values and (2) arranging them in some order. There are \( C(n, r) \) ways to pick the values and \( P(r, r) \) ways to order them, so \( P(n, r) = C(n, r) \times P(r, r) \) (and recall, \( P(r, r) = r! \)).

**Properties of \( C(n, r) \):**
- \( C(n, 0) = 1 \) for \( n \geq 0 \). (There is only one way to pick zero items from a set.)
- \( C(n, n) = 1 \) for \( n \geq 0 \). (There is only one way to pick all the items from a set.)
- \( C(n, r) = C(n, n-r) \) (Selecting \( r \) items from \( n \) is like removing \( n-r \) items from a set.)

**Questions**

1. BIC (Bowl of Ice Cream) serves bowls of ice cream with three scoops each. Scoops come in six flavors (boringly called \( A, B, C, D, E, \) and \( F \)), and a bowl must contain three different flavors.

1a. How many ways are there to serve three scoops if the order of scoop selection is important?

1b. Study the list below of all the ways to serve three scoops if we keep track of the order of scoops. Our eventual goal is to calculate the number of ice cream bowls possible if order is not important. We’ll do this by mapping 3-permutations to 3-combinations. What does the first column comprise? Why are there six rows? What in general does each column represent?

<table>
<thead>
<tr>
<th>ABC</th>
<th>ABD</th>
<th>ABE</th>
<th>ABF</th>
<th>ACD</th>
<th>...</th>
<th>BEF</th>
<th>CDE</th>
<th>CDF</th>
<th>DEF</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACB</td>
<td>ADB</td>
<td>AEB</td>
<td>AFB</td>
<td>ADC</td>
<td>...</td>
<td>BFE</td>
<td>CED</td>
<td>CFD</td>
<td>DFE</td>
</tr>
<tr>
<td>BAC</td>
<td>BAD</td>
<td>BAE</td>
<td>BAF</td>
<td>CAD</td>
<td>...</td>
<td>EBF</td>
<td>DCE</td>
<td>DCF</td>
<td>EDF</td>
</tr>
<tr>
<td>BCA</td>
<td>BDA</td>
<td>BEA</td>
<td>BFA</td>
<td>CDA</td>
<td>...</td>
<td>EFB</td>
<td>DEC</td>
<td>DFC</td>
<td>EFD</td>
</tr>
<tr>
<td>CAB</td>
<td>DAB</td>
<td>EAB</td>
<td>FAB</td>
<td>DAC</td>
<td>...</td>
<td>FBE</td>
<td>ECD</td>
<td>FCD</td>
<td>FDE</td>
</tr>
<tr>
<td>CBA</td>
<td>DBA</td>
<td>EBA</td>
<td>FBA</td>
<td>DCA</td>
<td>...</td>
<td>FBE</td>
<td>ECD</td>
<td>FDC</td>
<td>FED</td>
</tr>
</tbody>
</table>

1c. Obviously the number of rows \( \times \) the number of columns equals the total numbers of entries in the grid (the answer to 1a). Use that to determine the number of columns.

1d. How many different bowls of ice cream are possible where the order of scoops does not matter?
2. There are 100 people. Everyone shakes hands with everyone else. How many handshakes?

3. How many distinct 7-card hands can be drawn from a standard 52-card deck?

Some Review Questions for Section 5.5 (Counting Permutations)

4. How many 1-to-1 functions are there from the set of 5-bit strings to the set of 6-bit strings?

5a. Suppose a legal password is a sequence of 14 lower-case letters and/or digits, with no repeats?

5b. Suppose a legal password is a sequence of 14 lower-case letters and/or digits, with no repeats and the letters a, b, c, and d must appear in the password? (They can appear in any order, and they can appear in any 4 of the 14 positions in the password.) Try figuring out how many ways you can select 4 of the 14 characters and assign them a permutation of a, b, c, and d. Then multiply this by the number of ways you can assign the other 10 characters from the remaining characters (0–9, and e–z).

6. Remember the Dagwood sandwich from the last lecture? Between the two slices of bread, it had a permutation of 12 items with the restriction that the mustard had to be somewhere above the lettuce and the ketchup had to be somewhere below the lettuce. Try calculating the number of possible sandwiches as (the number of ways to select 3 positions for the ketchup, lettuce, and mustard) × (the number of ways to permute the remaining items into the 9 remaining positions).