CS 330 - Class 19, Tue Mar 30

Discrete Probability, pt 1

Chapter 7 - Discrete Probability

- Section 7.1 - Introduction to Discrete Probability
- Section 7.2 - Probability Theory
- Section 7.3 - Bayes' Theorem
- Section 7.4 - Expected Value and Variance

Probability

The extent to which an event is likely to occur, measured by the ratio of the favorable cases to the whole number of cases possible\(^1\)

- In 1700s's, Laplace defined probability of an event as number of successful outcomes divided by number of possible outcomes.
- Assumption: All outcomes are equally likely

\(^1\) (Apple dictionary)
Some Basic Terms

- **Experiment**: A procedure that yields one of a set of outcomes.
  - flipping a coin, rolling a die
- **Sample space** \( S \) = set of possible outcomes
  - \{H, T\}, \{face shows 1, 2, 3, 4, 5, 6\}
- **Event** \( E \) = a subset of the sample space
  - \{H\}, \{1,2\}
- **Probability of an event** \( p(E) = |E| / |S| \)
  - \( p(\{H\}) = |\{H\}| / |\{S\}| = 1/2 \)
  - \( p(\{1,2\}) = |\{1,2\}| / |\{1, 2, 3, 4, 5, 6\}| = 1/3 \)
  - \( p(\text{pick a club}) = 13/52 = 1/4 \)

Playing cards

- 52 cards: 4 suits \{♠, ♦, ♥, ♣\}, 13 kinds = ranks
  - Kinds: Ace, 2, 3, ..., 10, Jack, Queen, King
  - 53 cards if you include a Joker
- Probability of picking a club if you pick 1 card?
  - 13 clubs / 52 cards total = 1/4
- Probability of picking \( \geq 1 \) club if you pick 2 cards (with replacement)
  - \( |E| / |S| = 13^2 / 52^2 = 1/16 \) (which = \( (1/4)^2 \))
- Probability of picking a club and then a club (2 cards)?
- Product rule $13 \times 12 / 52 \times 52$ (w/o replacement) = 3/52

- Probability of picking exactly 1 club if you pick 2 cards w/repl?
- Order not important, so (club, -club) = (-club, club)
- 52 x 52 pairs, 13 x 39 x 2 of form (♣, -♣) or (-♣, ♣)
- $2 \times 13 \times 39 / 52 \times 52 = 3/8$

More Cards (all assuming standard 52-card deck)

Probability of having a full house after being dealt five cards?

- $|S| = C(52, 5) = 2,598,960$

- Full house = 3 of one kind, 2 of another
  - For triple: choose 1 of 13 ranks, 3 of 4 suits
  - For pair: choose 1 of 12 ranks, 2 of 4 suits
- $|E| = C(13, 1) \times C(4, 3) \times C(12, 1) \times C(4, 2) = 3744$

- $p$(full house) = $3744 / 2,598,960 \approx 0.00144 = 0.14\%$
More Cards

Probability of having two pair (after being dealt from cards from a standard deck)?

- 2 pair: Pick 2 of 13 kinds
- 2 of 4 suits for the 1st pair
- 2 of 4 suits for the 2nd pair
- 5th card: Pick 1 of 11 kinds, 1 of 4 suits

\[ | \text{two pair} | = C(13,2) \times C(4,2) \times C(4,2) \times C(11,1) \times C(4,1) = 123552, \quad |S| = C(52,5) \]

\[ p(\text{two pair}) = \frac{123552}{2,598,960} \approx 0.04753 \approx 4.75 \% \]

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Two pair - why aren't we multiplying

- \( C(13,1) \times C(4,2): \) 1 kind for 1st pair, then 2 suits for it
- \( C(12,1) \times C(4,2): \) 1 kind for 2nd pair, then 2 suits for it
- \( C(11,1) \times C(4,1): \) 1 kind for 3rd pair, then 1 suit for it

Because it doesn't matter in which order you pick the two pairs: AAKK. vs. KKAA.

So if we multiply the 3 terms above, we need to divide by 2! to remove the ordering

\[ \frac{\text{Big product}}{2!} = \frac{247104}{2!} = 123552 \text{ as before} \]
Why didn't this come up with the full house?

- There we were picking one suit for three cards and one
  for two cards, and those choices were independent of
  each other (pick 3 then 2 = pick 2 then 3).

- So product rule applied for |full house|

\[
\text{1st rank suits } C(13, 1) \times C(4, 3) \quad \text{2nd rank suits } C(12, 1) \times C(4, 2)
\]

Another ordering example

- Pick the 4 numbers 31, 25, 12, 42 from a collection of
  numbers 1 - 50.

- 31 .. 42 in that order?
  - \( |E| = 1, |S| = P(50, 4) = 50 \times 49 \times 48 \times 47 = 5,527,200 \), so
  \( p(31, 24, 12, 42) = 1/5,527,200 \)

- Different approaches, same result
  - In any order? \( |E| = 4! \) so multiply above by 4!,
    \( 4! / 5,527,200 = 1/230,300 \)

- Or \( |S| = C(50, 4) \) [all 4-nbr sets], \( E = 1 \) winning set

\[
p(...) = 1/C(50, 4) = 1 / (P(50, 4) / 4!) = 1/230,300
\]

\( S = \text{set of 4-nbr subsets} \)
**Mega-Millions Jackpot**

- To win
  - Pick five numbers $\in \{1-70\}$ (no duplicates, order unimportant) and also
  - One number $\in \{1-25\}$ (the "mega ball")
  - $|S| = \binom{70}{5} \times \binom{25}{1}$ [they choose 5 and 1 numbers]

- There's only one of the $\binom{70}{5}$ choices that wins and only one choice of the $\binom{25}{1}$ choices that wins
- So $|E| = 1$
- So $p(\text{mega-millions}) = 1 / \binom{70}{5} \times \binom{25}{1}$
  
  
  $= 1 / 12,103,014 \times 25 = 1/302,575,350$ (eek!)

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**Dice rolls**

- Standard die has six sides with 1 - 6 "pips" (= dots) on the six sides
- Fair die has $1/6$ chance of each side coming up.
- Rolling two dice: $S = \{1-6\} \times \{1-6\}$, size 36

- Probability of rolling "snake-eyes" = total of 2 when rolling two dice = rolling 1 on each
- Only one winning roll, (1,1)
- $p(\text{snake-eyes}) = 1 / |S| = 1/36$
  
  
  $\text{Same probability for rolling a } 12 = 6 + 6$
• Probability of rolling a 7
  • $E = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$ size 6
  • $p(\text{rolling 7}) = 6/36 = 1/6$

• Probability of not rolling a 7
  • $p(\text{not 7}) = |S - E| / |S| = 1 - |E| / |S|$
  • $p(\text{not 7}) = 1 - p(\text{rolling 7}) = 5/6$

\[\text{prob rolling 4} \quad E = \{(1,3), (2,2), (3,1)\}\]

\[p(4) = 3/36 = 1/12\]

\[\text{prob rolling 3} \quad E = \{(1,2), (2,1)\}\]

\[p(3) = 2/36 = 1/18\]

\[\left\{\begin{array}{c}
\text{Any roll} \\
\text{roll a 7}
\end{array}\right.\]

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**Some Rules**

• **Product rule:** $p(E_1 \times E_2) = p(E_1) \times p(E_2)$

\[E_1 \times E_2 = |E_1/| \times |E_2|\]

• **Sum rule:**
  • If $E_1$ and $E_2$ are independent: $p(E_1 \cap E_2) = 0$
  • then $p(E_1 \cup E_2) = p(E_1) + p(E_2)$
  • Otherwise $p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$

• **Compliment rule:** $p(\overline{E}) = 1 - p(E)$
  • Event $\overline{E} = S - E$ is complement of $E$

\[p(\text{roll } \neq 7) = 1 - \text{prob (roll 7)}\]

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