Last time

- Section 7.1 - Introduction to Discrete Probability
- Laplace: \( p(E) = \frac{|E|}{|S|} \) if each event in \( S \) is equally likely
- \( p(\text{kind of 5-card hand}) = \frac{\# \text{ cards of that kind}}{\binom{52}{5}} \)
- \( p(\text{kind of dice roll}) = \frac{\# \text{ relevant dice rolls}}{\text{all dice rolls}} \)
- \( p(\text{lottery win}) = \frac{\# \text{ winning picks}}{\# \text{ all possible picks}} \)

Rules (review)

- **Product rule:** \( p(E_1 \times E_2) = p(E_1) \times p(E_2) \)
- One event followed by another event
- **Sum rule:**
  - \( p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2) \)
  - Simplifies if \( E_1 \) and \( E_2 \) are independent: \( p(E_1 \cap E_2) = 0 \)
- One event or another event
- **Compliment rule:** \( p(E) = 1 - p(\overline{E}) \)
  - Event \( \overline{E} = S - E \) is complement of \( E \)
  - The opposite event from \( E \)
Subtleties (review)

- For 2 pair of cards, we must ignore which kind of pair gets chosen first.
  - \(|2 \text{ pair}| = C(13,2) \times C(4,2) \times C(4,2) \times C(11,1) \times C(4,1)\)
  - Choose (say) Aces & Fives, Club & Heart (aces), Diamond and Spade (fives), and fifth card
  - Un-permute (Aces then Fives) vs (Fives then Aces)
- For full house (3 of suit\(_1\), 2 of suit\(_2\)) choices commute:
  - \(C(13,3) \times C(4,1) \times C(12,2) \times C(3,1)\) equals
  - \(C(13,2) \times C(4,1) \times C(12,3) \times C(4,1)\)
- Pick a collection of values
- Ordered values? Unordered values?

Today, start with Monty Hall Problem

- Subtlety: Must analyze the whole picture.
- Game show where one of three doors hides a prize.
  - You choose one door.
  - Monty opens one of the two remaining doors and shows there's nothing.
  - Monty gives you the opportunity of staying with the door you have or switching to the remaining closed door.
- Should you?
Reasonable but flawed analysis

- There were three doors, but one's been shown to fail, so there are only two doors left.
- One of them hides the prize and one doesn't.
- So the odds are even, so it doesn't matter.

- The flaw is in not looking at the history of the whole experiment.

Correct analysis of Monty Hall Problem

- At the beginning, you picked one of three doors.
  - Probability you picked the right door: $1/3$.
  - Probability you picked the wrong door: $2/3$.
  - Monty showed you an empty door.
  - If you picked the right door, you should stay with it. \( \text{Probability} = 1/3 \).
  - If you picked the wrong door originally, then the right door is the one that's left over: \( \text{Probability} = 2/3 \).
  - You should switch !!
Deal or No Deal — Similar situation

- There are 10 suitcases, 9 empty, 1 with $1,000,000.
- You pick a suitcase.
- The host repeatedly opens up one of suitcases to show it's empty. Keeps going until there are 2 suitcases left.
- At any point, you can pick a suitcase and leave with it.
- How long should you wait, and what should you do then?

Deal or No Deal

- This is the Monty Hall problem on steroids.
- Probability you picked the right suitcase to start: 1/10
- Probability that you picked the wrong suitcase: 9/10
- At the beginning, there was a 90% chance that the money was in one of the other suitcases.
- Opening an empty suitcase doesn't change the probability you picked the wrong suitcase to start.
- What's changed is that if you picked the wrong suitcase, then the right one is one of the fewer and fewer suitcases in front of you. Probability 9/10
- Wait until there are 2 suitcases and then switch !!!
What if not all outcomes are equally likely?

- Laplacian definition doesn't apply
- Instead, for each outcome \( x \in E \), you need to know \( p(x) \), and \( p(E) = \Sigma_{x \in E} p(x) \).
- The function \( p: S \rightarrow \mathbb{R} \) is the **probability distribution**.
- Ideally \( p(x) = \) limit of \( \frac{\# \ x \ occurs}{\# \ experiment \ trials} \) as \( \# \ trials \rightarrow \infty \)

**Assumptions**

- A limit exists.
- The result of an experimental trial doesn't depend on past history of trials
- And for us, \( S \) is countable (finitely or infinitely)

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**Example**

- **Unfair die:** Has six faces but one (say, the 6) is twice as likely to come up as any of the five other faces.
  - \( p(\text{roll 6}) = 2x \) where \( x = p(\text{roll x}) \) for \( x = 1, 2, \ldots, 5 \)
  - \( 1 = \text{sum of all probabilities} = 2x + 5x = 7x \)
  - \( p(\text{roll 1}) = p(\text{roll 2}) = \ldots = p(\text{roll 5}) = x = 1/7 \approx 14.2\% \)
  - \( p(\text{roll 6}) = 2/7 \approx 28.5\% \)

- **Really unfair die:** 6 is twice as likely to come up than not.
  - \( p(\text{roll 6}) = 2x, p(\text{roll 1 or 2 or ... or 5}) = x \)
  - \( 1 = \text{sum of probabilities} = 2x + x, \text{so } x = 1/3 \)
  - Rolling 6 happens 2/3 of the time
  - For all other \( n \), rolling \( n \) has probability 1/15
Uniform Distribution

- Laplace assumed each $x \in S$ is equally likely:
  \[ p(x) = \frac{1}{n} \quad (n = |S|) \quad (\text{and } S \text{ finite}) \]
- This is called the the \textit{uniform distribution}.
- \textit{Selecting at random}: selecting under a uniform distribution.
- Uniform distribution requires that probabilities of events don't affect each other.

Conditional Probability

- But sometimes events are connected.
- The \textit{conditional probability of $E$ given $F$} is the probability of event $E$ occurring given that event $F$ has already occurred.
  - It's defined as the probability of $E$ \textit{and} $F$ occurring divided by probability of $F$ \textit{by itself}.
- \textbf{Definition}: \[ p(E \mid F) = \frac{p(E \cap F)}{p(F)} \]
  - Assumption $p(F) > 0$. 
Example of Non-Conditional and Conditional Probabilities

- The non-conditional example first:
  - \( S = \{0000, 0001, 0010, \ldots, 1111\} \), bitstrings of length 4
  - For one \( x \in S \), what is \( p(\text{x has} \geq 2 \ \text{consecutive 0s}) \)?
    - \( P(4,4) + P(4,3) + 3 = (1 + 4 + 3) = 8 \) such bitstrings
  - \( P(4,4) \) strings with four 0s
  - \( P(4,3) \) strings with three 0s (0001, 0010, 0100, 1000)
  - 3 strings with 2 adjacent 0s (0011, 1001, 1100)
  - \( p(\text{x with} \geq 2 \ \text{consecutive 0s}) = 8 / 2^4 = 8/16 = 1/2 \)

Conditional Probability

- Now say we've already picked the leftmost bit of our string and it's a 0. (Event F: bit 3 of string = 0)
  - Now how many strings have \( \geq 2 \) consecutive 0s?
    - 0000, 0001, 0010, 0100, 0011 (omit 1000, 1001, 1100)
  - \( p(E \cap F) = 5/16 \), \( p(F) = 8/16 \), so \( P(E|F) = 5/16 / 8/16 = 5/8 \)

How about \( P(E|\bar{F}) \)?
\[
\frac{3/16}{8/16} = \frac{3}{8}
\]

\( E \) strings:
- 0000, 0001, 0010, 0100, 0011, 1000, 1100
• Since we picked a 0 for bit 3
  • We’ve avoided picking the eight strings \( \subseteq S \) beginning with 1, which made us lose three strings meeting the criterion
  • And of the eight strings beginning with 0, five meet the criterion.

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**Conditional Probability Example**

• We have a drawer with blue and red socks.
  • Equally likely to pick a blue or a red sock.
  • We’ve already picked a blue sock.
  • If we pick another sock, what's the conditional probability that we'll have 2 blue socks?

\[
\begin{align*}
E & = \{BBB, BBR, BBK, BRR, RBB, RBR, RRK, RRR\} \\
S & = \{BBB, BBR, BBK, BRR, RBB, RBR, RRK, RRR\} \\
F & = \{BBB, BBR, BBK, BRR, RBB, RBR, RRK, RRR\} \text{ 1st sock blue} \\
E & = \{BBB, BBR, BBK, BRR, RBB, RBR, RRK, RRR\} \text{ 4th sock blue}
\end{align*}
\]
• (Conditional probability of 2 blue socks given first sock was blue)
  
  • Sample space: Let $S = \{BB, BR, RB, RR\}$.
  
  • Event: 1st sock blue. Let $F = \{BB, BR, RB\}$, $p(F) = \frac{3}{4}$
  
  • Event: 2 blue socks. Let $E = \{BB\}$, $p(E) = \frac{1}{4}$
  
  • $E \cap F = \{BB\}$, $p(E \cap F) = \frac{1}{4}$
  
  • $p(E \mid F) = \frac{p(E \cap F)}{p(F)} = \frac{(\frac{1}{4})}{(\frac{3}{4})} = \frac{1}{3}$

  $S = BB, BR, RB, RR$
  
  $p(E) = \frac{1}{4}$  $E = BB$
  
  $p(F) = \frac{3}{4}$  $F = BB, BR$

$\frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$

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**Independence**

• If $p(F)$ doesn't affect $p(E \mid F)$, then $E$ and $F$ are independent events.

• $E$ occurring is not influenced by $F$ having occurred

• In general, $p(E \mid F) = \frac{p(E \cap F)}{p(F)} = \frac{p(E)}{p(F)}$

• For $E$ and $F$ to be independent, $p(E \mid F) = p(E)$.

• $p(E \cap F) = p(E \mid F) \cdot p(F) = p(E) \cdot p(F)$

• Another characterization:

• For $E$ and $F$ to be independent, $p(E \cap F) = p(E) \cdot p(F)$.  


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Back to Bitstring Example

- For randomly generated bitstrings of length 4
- \( E = \) Bitstring begins with a 1.
- \( F = \) Bitstring has even # of 1s

- Are \( E \) and \( F \) independent?
  - \(|S| = 16, |E| = 8, |F| = 8\)
  - \( p(E) = 8/16 = 1/2, p(F) = 8/16 = 1/2\)
  - \( p(E \cap F) = \{1001, 1010, 1100, 1111\} \mid / 16 = 1/4\)
  - \( p(E) \times p(F) = 1/2 \times 1/2 = 1/4 = p(E \cap F)\)
  - \( E \) and \( F \) are independent

(A different) Red/Blue Socks Example

- Drawer has red and blue socks, equal probability of picking red or blue. We pick 3 socks
  - Event \( E = \) We have \( \geq 1 \) red and \( \geq 1 \) blue sock
  - Event \( F = \) We have \( \leq 1 \) blue sock
  - Are \( E \) and \( F \) independent?
8. \( S = \{BBB, BBR, BRB, BRR, RBB, RBR, RRB, RRR\} \)

6. \( E = \{BBR, BRB, BRR, RBB, RBR, RRB\} \)

4. \( F = \{BRR, RBR, RRB, RRR\} \)

3. \( E \cap F = \{BRR, RBR, RRB\} \)

- \( p(E) \times p(F) = (3/4) \times (1/2) = 3/8 = p(E \cap F) \)

- Yes, \( E \) and \( F \) are independent

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**Pairwise Independence and Mutual Independence**

- The events \( E_1, E_2, \ldots, E_n \) are **pairwise independent** if for all \( i \neq j \), \( E_i \) and \( E_j \) are independent.
- I.e., \( p(E_i \cap E_j) = p(E_i) \times p(E_j) \)
- They are **mutually independent** if every subset (size \( \geq 2 \)) is independent in this sense.

*Cover this next time*