Solutions to Homework Assignment 2
CS 330 Discrete Structures
Spring Semester, 2019

Solution:

1. For each variable, we have 2 possible cases: True, False. With \( n \) variables we have \( 2^n \) lines in one truth table. Then for each line in the truth table, we could have two choices, True or False, depending on the proposition. So totally we have \( \prod_{i=1}^{2^n} 2 = 2^{2^n} \) different truth tables.

2. We have five 0s and 14 1s. Since every 0 must be immediately followed by two 1s, we require five blocks to put 011s and the remaining 4 1s can be put in any four position between 011s. Totally we need 9 blocks to put 5 011s and 4 1s. Since the order of the same bits is not important, there could be:

\[
C(9, 5) = \frac{9!}{5!(9-5)!} = 126
\]

bit strings.

3. Based on the definition of binomial coefficient, we can easily find that the both sides of the equality are equal to the following term:

\[
\frac{(n-1)!n!(n+1)!}{(k-1)!k!(k+1)!(n-k-1)!(n-k)!(n-k+1)!}
\]

It’s called the hexagon identity because these six binomial coefficients make the vertices of a regular hexagon in Pascal’s triangle.

4. (a) using a combinatorial argument:
\( \binom{2n}{2} \) means pick 2 from \( 2n \) choices. This is essentially equal to the addition of the following three cases:
   - pick 2 from the first half, which is \( \binom{n}{2} \).
   - pick 2 from the other half, which is \( \binom{n}{2} \).
   - pick 1 from the first half, and pick 1 from the other half, which is \( \binom{n}{1} \times \binom{n}{1} = n^2 \).

The addition is \( 2\binom{n}{2} + n^2 \).

(b) by algebraic manipulation:
\[
\binom{2n}{2} = \frac{2n!}{2!(2n-2)!} = \frac{2n(2n-1)}{2} = n(2n-1) = n(n-1) + n^2
\]
\[
= 2 \frac{n(n-1)}{2} + n^2
\]
\[
= 2 \binom{n}{2} + n^2
\]
(c) prove by induction:

1. **Base case**
   when \( n = 1 \), LHS = 1, RHS = 1. Therefore, LHS = RHS.

2. **Inductive Hypothesis**
   Assume when \( n = k \) the equality holds. That is, we assume:
   \[
   \binom{2k}{2} = 2 \binom{k}{2} + k^2
   \]

3. **Inductive Step & Proof**
   When \( n = k + 1 \),
   \[
   \binom{2k + 2}{2} = \binom{2k + 1}{1} + \binom{2k + 1}{2} \quad \text{(Pascal’s Identity)}
   
   = 2k + 1 + \binom{2k}{1} + \binom{2k}{2} \quad \text{(Pascal’s Identity)}
   
   = 2k + 1 + 2k + 2 \binom{k}{2} + k^2 \quad \text{(Inductive Hypothesis)}
   
   = 2(k + \binom{k}{2}) + (k + 1)^2
   
   = 2\binom{k}{1} + \binom{k}{2} + (k + 1)^2
   
   = 2\binom{k + 1}{2} + (k + 1)^2 \quad \text{(Pascal’s Identity)}
   
   Hence we proved the equality holds when \( n = k + 1 \). Combining the Base Case, Inductive Hypothesis and the Inductive Step & Proof, we can deduce that for any integer \( n \geq 1 \), the equality holds.

5. **a.** Since both the balls and boxes are labeled, the order matters. We can apply the rule of product. For the first ball, we have 7 choices. For the second one, we have 6 choices, and so forth. Then, we have \( 7 \times 6 \times 5 \times 4 \times 3 = 2520 \) ways.

   **d.** Since both the balls and boxes are unlabeled, no matter how we put the balls in the boxes, all the ways are identical. Therefore, there is only one way.

6. **(a)**
   \[
   \binom{k}{2} = \frac{k(k-1)}{2} = \frac{k(k-1)}{2} + (\frac{k(k-1)}{2} - 1)
   
   = 3\frac{(k + 1)k(k - 1)(k - 2)}{4 \times 3 \times 2}
   
   = 3\frac{k + 1}{4}
   
   \]

   **(b)** LHS of the equality is the number of different ways to choose 2 pairs among the pairs of \( k \) items where two items in each pair should be different.
   Following the hint, we consider adding the \( k + 1 \)-st element “DUP” into the existing \( k \) items. If we pick 4 from these \( k + 1 \) items, we have two different cases depending on whether “DUP” is picked or not.
If there is no “DUP” in these 4 items, firstly we choose 4 from \( k \) valid items and pick 0 from the “DUP”, which leads to \( \binom{k}{4} \) ways to do it. Then, we have \( \binom{4}{2} \cdot \binom{2}{2}/2 = 3 \) ways to have 2 pairs. The reason why we should divide 2 can be explained via an example. Suppose we have 4 numbers 1,2,3 and 4. When we pair them into 2 pairs, pairing (1, 2) as a first pair and (3, 4) as a second pair is identical to pairing (3, 4) as a first pair and (1, 2) as a second pair. However, in our term \( \binom{4}{2} \cdot \binom{2}{2} \), this duplicity is not reflected. Therefore, there are \( 3 \binom{k}{4} \) ways in this case.

If one of them is the “DUP”, which is choosing 3 from \( k \) valid items and 1 from “DUP” \( \binom{k}{3} \), any one of other 3 items can be duplicated to make the items into 4 items. In this case, we have 3 choice to make the duplication at the first step. After the duplication, since we have two duplicate items, the only way to pair them into two pairs is to put two duplicate item into each pair, which leads to only one way to pair these 4 items. So, there are \( 3 \binom{k}{3} \) ways in this case.

The addition of the above two cases is equivalent to choosing 2 pairs among the pairs of \( k \) items. By the rule of sum, the number of final possibilities is \( 3 \binom{k}{4} + 3 \binom{k}{3} = 3 \binom{k+1}{4} \) (Pascal’s Identity). Therefore, the equality \( \binom{4}{2} = 3 \binom{k+1}{4} \) holds.