1. For simplicity, denote the probability that no one has the same birthday when there are \( n \) people as \( p(n) \). Then, what we want is

\[
1 - p(n) \geq 0.5 \Rightarrow p(n) \leq 0.5
\]

\[
p(n) = \frac{\# \text{ of ways to have different birthday for everyone}}{\# \text{ of ways of total assignments}}
\]

\[
= \frac{260!}{(260-n)!} \cdot \frac{259}{260} \cdot \frac{258}{260} \cdots \frac{260-n+2}{260} \cdot \frac{260-n+1}{260} \leq 0.5
\]

Solving the above inequality leads to \( n \geq 20 \). Therefore we need 20 people.

2. a) \( \binom{49}{6} \), because the order does not matter.

b.i) \( \left( \frac{1}{\binom{49}{6}} \right)^2 \), because the chance we have one specific set of 6 numbers is \( \frac{1}{\binom{49}{6}} \), and it happened twice.

b.ii) For this, we can first calculate the probability that every drawing is unique (i.e., no identical combinations) out of \( n \) combinations, which is:

\[
\frac{\binom{49}{6}}{\binom{49}{6}} \cdot \frac{(\binom{49}{6}-1)}{\binom{49}{6}} \cdot \frac{(\binom{49}{6}-2)}{\binom{49}{6}} \cdots \frac{(\binom{49}{6})-n+1}{\binom{49}{6}} = \frac{(\binom{49}{6})!}{((\binom{49}{6})-n)!} \cdot \frac{\binom{49}{6}}{\binom{49}{6}}^n
\]

Then, the probability that some pair have been the same for \( n \) drawings is

\[
1 - \frac{(\binom{49}{6})!}{((\binom{49}{6})-n)!} \cdot \frac{\binom{49}{6}}{\binom{49}{6}}^n
\]

which must be about 50%. With this equation, \( n \) can be solved easily by a program.

b.iii) Following the same theory as b.ii), we can first calculate the probability that every adjacent drawing is different for \( n \) drawings, which is:

\[
\frac{\binom{49}{6}}{\binom{49}{6}} \cdot \frac{(\binom{49}{6})-1}{\binom{49}{6}} \cdot \frac{(\binom{49}{6})-2}{\binom{49}{6}} \cdots \frac{(\binom{49}{6})-(1-n)}{\binom{49}{6}} \cdot \frac{1}{\binom{49}{6}} = \left(1 - \frac{1}{\binom{49}{6}}\right)^{n-1}
\]

because every drawing only excludes one combination (the one appeared in the previous drawing). We assumed \( n \) is even without the loss of generality. Then, the probability that some adjacent pair are same is:

\[
1 - \left(1 - \frac{1}{\binom{49}{6}}\right)^{n-1}
\]

which must be about 50%. This equation can be solved easily by hand, and \( n \) is 9,692,844.
3. a) Since the coins are fair, the probability of showing head is \( \frac{1}{2} \) and the one of showing tail is also \( \frac{1}{2} \).

Then, the probability of student \( i \)'s coin showing head and \( n-1 \) showing tails is

\[
\frac{1}{2} \left( \frac{1}{2} \right)^{n-1} = \frac{1}{2^n}
\]

The probability of student \( i \)'s coin showing tail and \( n-1 \) showing heads is also \( \frac{1}{2^n} \) due to the same reason. Then, according to the rule of sum, the probability of an odd toss happening at some student \( i \)'s coin is

\[
\frac{1}{2} \left( \frac{1}{2} \right)^{n-1} + \frac{1}{2} \left( \frac{1}{2} \right)^{n-1} = \frac{1}{2^n}
\]

Since there are \( n \) students, the probability of an odd toss for any student is (according to the rule of sum)

\[
\frac{1}{n} \cdot \frac{1}{2^{n-1}} = \frac{1}{2^{n-1}}
\]

Therefore, \( p = \frac{n}{2^{n-1}} \).

Also, ending on the \( k \)-th round means there were more than one odd toss in the previous \( k-1 \) rounds and one odd toss appeared at the \( k \)-th round, whose probability is:

\[(1-p)^{k-1}p\]

Therefore, the expected number of the rounds is:

\[
\sum_{i=1}^{\infty} i \cdot (1-p)^{i-1}p = p \sum_{i=1}^{\infty} i(1-p)^{i-1} = p \cdot \frac{1}{(1-(1-p))^2} = \frac{1}{p} = \frac{2^{n-1}}{n}
\]

b) Suppose the probability that the cheating student's coin shows head is \( p(x) \). Then, the false coin shows tail by a probability \( 1 - p(x) \). Then, in this case, the probability that a student \( i \)'s (including cheating student) coin is an odd toss is

\[
\frac{1}{2^{n-1}}p(x) + \frac{1}{2^{n-1}}(1-p(x)) = \frac{1}{2^{n-1}}
\]

Therefore, the probability \( p \) remains same, and thus the answer does not change.

4. a) First of all, they must have at least two children to stop, which is obvious. The probability that the first child is a boy and then they stop having children after having the \( k \)-th child is

\[p^{k-1}(1-p)\]

The probability that the first child is a girl and then they stop having children after having the \( k \)-th child is

\[(1-p)^{k-1}p\]

Then, the probability that they will stop having children after having the \( k \)-th child is

\[p^{k-1}(1-p) + (1-p)^{k-1}p\]
which leads to the expected number of children

\[
\sum_{i=2}^{\infty} i(p^{i-1}(1-p) + (1-p)^{i-1}p) \\
= \sum_{i=2}^{\infty} ip^{i-1}(1-p) + \sum_{i=2}^{\infty} i(1-p)^{i-1}p \\
=(1-p)\sum_{i=2}^{\infty} ip^{i-1} + p\sum_{i=2}^{\infty} i(1-p)^{i-1} \\
=(1-p)\left(\frac{1}{(1-p)^2} - 1\right) + p\left(\frac{1}{p^2} - 1\right) \\
= \frac{1}{1-p} - (1-p) + \frac{1}{p} - p = \frac{p+1-p}{p(1-p)} - 1 = \frac{1-p+p^2}{p(1-p)}
\]

This answer does not make sense when \( p = 0 \) or \( p = 1 \) because the denominator cannot be 0.

If \( p = 0 \) or \( p = 1 \), the couple will never stop having babies since they will always have same sex children.

b) Again, they must have at least two children to stop. The probability that the first child is a boy and then they stop having children after the \( k \)-th child is

\[
p(1-p)^{k-2}p
\]

The probability that the first child is a girl and then they stop having children after the \( k \)-th child is

\[
(1-p)p^{k-2}(1-p)
\]

Then, the total probability that they stop having children after the \( k \)-th child is

\[
p^2(1-p)^{k-2} + (1-p)^2p^{k-2}
\]

which leads to the expected number of children

\[
\sum_{i=2}^{\infty} i(p^2(1-p)^{i-2} + (1-p)^2p^{i-2}) \\
= \sum_{i=2}^{\infty} i \cdot p^2(1-p)^{i-2} + \sum_{i=2}^{\infty} i(1-p)^2p^{i-2} \\
= \frac{p^2}{1-p} \sum_{i=2}^{\infty} i(1-p)^{i-1} + \frac{(1-p)^2}{p} \sum_{i=2}^{\infty} ip^{i-1} \\
= \frac{p^2}{1-p} \left(\frac{1}{p^2} - 1\right) + \frac{(1-p)^2}{p} \left(\frac{1}{(1-p)^2} - 1\right) \\
= \frac{1}{1-p} - \frac{p^2}{1-p} + \frac{1}{p} - \frac{(1-p)^2}{p} \\
= \frac{1-p^3 - (1-p)^3}{p(1-p)} \\
= \frac{1 - p^3 - (1 - 3p + 3p^2 - p^3)}{p - p^2} \\
= \frac{3p - 3p^2}{p - p^2} = 3
\]
This answer does not make sense either when $p = 0$ or $p = 1$ because the denominator cannot be 0. If $p = 0$ or $p = 1$, they will have only two children.

5. *“The probability of creating the perfect ..... less than 1 chance in 9.2 quintillion chances.”*

Since there are 64 teams in the tournament, there are 63 games in total ($32+16+8+4+2+1$). In order to get a perfect match, one needs to predict every result correctly. Suppose the accuracy of one’s prediction is 50%, then getting a perfect match yields the probability $\frac{1}{2^{63}} = 9,223,372,036,854,775,808$ which is one over 9.2 quintillion.

*“It’s more likely that the Chicago Cubs and the Chicago White Sox will win the next 16 World Series games.”*

The chance that either Chicago Cubs or Chicago White Sox will win one World Series game is $\frac{2}{30}$ since there are 30 teams. Then, the chance that the winner of the World Series is one of them for 16 times is $\left(\frac{2}{30}\right)^{16} = \frac{1}{6,568,408,355,712,890,625} \approx 1.6 \times 10^{-9}$ which is greater than 9.2 quintillion shown above.

*“If you want to sit down and ....flip heads 53 times in a row.”*

**Typo: 53 should have been 63.**

The chance of having 63 times of heads in a row is $\frac{1}{2^{63}}$, about one in 9.2 quintillion.

*“Suppose you know that a No.1 seed ... would still be only about 1 in 128 billion with those odds.”*

If one knows a higher-seed team will always win over a lower-seed team, he can exactly predict the result of those games. 1 in 128 billion is approximately equal to $\frac{1}{2^{37}}$ (the professor in Depaul Univ. approximated the billion ($10^9$) as $2^{30}$ because $2^{10} \approx 10^3$), so the professor in Depaul Univ. may have intended to say with the knowledge we would predict the results of 26 games correctly.

*“Additionally, the chances of ... first round is about 1 in 17,000.”*

Similarly, 1 in 17,000 is approximately equal to $\frac{1}{2^{17}}$, implying the professor intended to say with the knowledge one would predict the results of 14 games correctly.