Solutions

1. (15 pts)
   (a) (5 pts)
   According to the problem, we have that
   
   \[ \Pr(\text{No two birthdays on same stem}) = \frac{(10!)/(10-n)!}{10^n} \]

   Thus, for the complementary event, there is
   
   \[ \Pr(\text{Birthdays on the same stem}) \equiv \Pr(E_1) = 1 - \Pr(\text{No two birthdays on same stem}) \]
   \[ = 1 - \frac{10 \cdot 9 \cdot 8 \cdots (10 - n + 1)}{10^n} \]

   Namely, we want to find the \( n \) that makes
   \[ \Pr(E_1) \geq 0.5 \implies \frac{10 \cdot 9 \cdots (10 - n + 1)}{10^n} \leq 0.5 \]

   Thus, calculate every \( n \) from 1, we find that \( n = 5 \).

   (b) (5 pts)
   Similar to the previous problem, we have that
   
   \[ \Pr(\text{No two birthdays on same branch}) = \frac{(12!)/(12-n)!}{12^n} \]
Thus, for the complementary event, there is

\[ \Pr(\text{Birthdays on the same branch}) \equiv \Pr(E_2) \]
\[ = 1 - \Pr(\text{No two birthday on same branch}) \]
\[ = 1 - \frac{12 \cdot 11 \cdot 10 \cdots (12 - n + 1)}{12^n} \]

Namely, we want to find the \( n \) that makes

\[ \Pr(E_2) \geq 0.5 \implies \frac{12 \cdot 11 \cdots (12 - n + 1)}{12^n} \leq 0.5 \]

Thus, calculate every \( n \) from 1, \( n_2 = 5 \).

(c) (5 pts)
Similar to the previous problems, we have that

\[ \Pr(\text{No two birthdays on same Chinese name}) = \frac{(60!)/(60 - n)!}{60^n} \]

Thus, for the complementary event, there is

\[ \Pr(\text{Birthdays on the same Chinese name}) \equiv \Pr(E_3) \]
\[ = 1 - \Pr(\text{No two birthday on same Chinese name}) \]
\[ = 1 - \frac{60 \cdot 59 \cdot 58 \cdots (60 - n + 1)}{60^n} \]

Namely, we want to find the \( n \) that makes

\[ \Pr(E_3) \geq 0.5 \implies \frac{60 \cdot 59 \cdots (60 - n + 1)}{60^n} \leq 0.5 \]

Thus, calculate every \( n \) from 1, \( n_3 = 10 \).

2. Page 468 #32. (15 pts)
Let \( E_1 \) indicate the event that bit string begins with ‘1’, and \( E_2 \) indicates the event that the string ends with ‘00’. Thus, the probability we want to get is \( \Pr(E_1 \cup E_2) \).
(a) (5 pts) a 0 bit and a 1 bit are equally likely.
In this case, $\Pr(E_1) = 1/2$ and $\Pr(E_2) = 1/2 \cdot 1/2 = 1/4$. So,
\[
\Pr(E_1 \cup E_2) = \Pr(E_1) + \Pr(E_2) - \Pr(E_1) \cdot \Pr(E_2) \\
= \frac{1}{2} + \frac{1}{4} - \frac{1}{2} \cdot \frac{1}{4} \\
= \frac{5}{8}
\]

(b) (5 pts) the probability that a bit is a 1 is 0.6.
In this case, $\Pr(E_1) = 0.6$ and $\Pr(E_2) = 0.4 \cdot 0.4 = 0.16$. So,
\[
\Pr(E_1 \cup E_2) = \Pr(E_1) + \Pr(E_2) - \Pr(E_1) \cdot \Pr(E_2) \\
= 0.6 + 0.16 - 0.6 \cdot 0.16 = 0.664
\]

(c) (5 pts) the probability that the $i$th bit is a 1 is $1/2^i$ for $i = 1, 2, 3, \ldots, 10$.
In this case, $\Pr(E_1) = \frac{1}{2}$ and $\Pr(E_2) = (1 - \frac{1}{2^9}) \cdot (1 - \frac{1}{2^{10}})$. So,
\[
\Pr(E_1 \cup E_2) = \Pr(E_1) + \Pr(E_2) - \Pr(E_1) \cdot \Pr(E_2) \\
= 1 - \frac{1}{2^9} + \frac{1}{2^{11}} + \frac{1}{2^{19}} - \frac{1}{2^{20}}
\]

3. Page 477 #22. (15 pts)

(a) (5 pts) $p(S) = \frac{s}{s + h}$ and $p(\bar{S}) = \frac{h}{s + h}$.

(b) (10 pts) Let $E$ indicate the event that a message includes word $w$. According to the problem, there is $p(E|S) = p(w)$ and $p(E|\bar{S}) = q(w)$.

Applied Bayes’ theorem here, we can get
\[
p(S|W) = \frac{p(W|S)p(S)}{p(W|S)p(S) + p(W|\bar{S})p(\bar{S})} \\
= \frac{p(w)\frac{s}{s + h}}{p(w)\frac{s}{s + h} + q(w)\frac{h}{s + h}} \\
= \frac{p(w)s}{p(w)s + q(w)h}
\]
4. Page 493 #40. (15 pts)

Firstly, the linear search algorithm is shown as following.

Algorithm 1 Linear search.
for $i \leftarrow 1$ to $n$ do
    if $i^{th}$ element equals to $x$ then
        return $i$
    end if
end for
return None

From this algorithm, we find that for every element in the list, we need to do 2 times comparisons, one for whether $i \leq n$ and the other one for whether $i^{th}$ element equals to $x$. So, there are $2n + 1$ comparisons for $x$ is not in the list, and $2i$ comparisons for $x$ is in the list and at $i^{th}$ position.

According to the problem, we get

$$
\Pr(x \text{ not in the list}) = 1 - \Pr(x \text{ in the list}) = 1 - \sum_{i=1}^{n} \frac{i}{n(n+1)} = 1/2
$$

Thus, the expected value of comparisons should be

Expected value = $\Pr(x \text{ not in the list}) \cdot (2n + 1) + \sum \Pr(x \text{ at position } i) \cdot (2i)$

$$
= \frac{1}{2} \cdot (2n + 1) + \sum_{i=1}^{n} 2i \cdot \frac{i}{n(n+1)}
= \frac{2n + 1}{2} + \frac{2}{n(n+1)} \cdot \sum_{i=1}^{n} i^2
= \frac{2n + 1}{2} + \frac{2}{n(n+1)} \cdot \frac{n(n+1)(2n+1)}{6}
= \frac{10n + 11}{6}
$$
5. (20 pts)

(a) (10 pts)
First, \( p = \frac{2 \cdot \binom{n}{1}}{2^n} = \frac{n}{2^{n-1}} \).

Moreover, if the tossing ends at \( k \)th round, then the previous \( k - 1 \) rounds must have no odd and \( k \)th must be odd. Namely, the probability of ending on the \( k \)th round should be \((1 - p)^{k-1}p\).

Thus, the expected ending round should be

\[
\text{Expected value} = \sum_{i=1}^{\infty} i \cdot (1 - p)^{i-1} \cdot p
\]

\[
= p \cdot \sum_{i=1}^{\infty} i \cdot (1 - p)^{i-1}
\]

\[
= p \cdot \frac{1}{(1 - (1 - p))^2}
\]

\[
= \frac{1}{p} = \frac{2^{n-1}}{n}
\]

(b) (10 pts)
According to the problem, the cheating coin only affects the probability \( p \) that odd occurred in a round. Assuming the cheating coin has probability \( p_c \) to get head. The new probability \( p' \) that odd occurred should be

\[
p' = p_c \cdot \binom{n-1}{1} + 1 - p_c \cdot \binom{n-1}{1} + p_c + (1 - p_c) = \frac{n}{2^{n-1}} = p
\]

Thus, the expected number of ending rounds doesn’t change.
6. (20 pts)

(a) (10 pts)
According to the problem, the expected ending round should be

Expected value = \[ \sum_{i=2}^{\infty} i \cdot ((1-p)^{i-1} \cdot p + p^{i-1} \cdot (1-p)) \]

= \[ \sum_{i=2}^{\infty} i \cdot (1-p)^{i-1} p + \sum_{i=2}^{\infty} i \cdot p^{i-1} (1-p) \]

= \[ p \cdot \left( \sum_{i=1}^{\infty} i \cdot (1-p)^{i-1} - 1 \cdot (1-p)^{1-1} \right) + (1-p) \cdot \left( \sum_{i=1}^{\infty} i \cdot p^{i-1} - 1 \cdot p^{1-1} \right) \]

= \[ p \cdot \left( \frac{1}{p^2} - 1 \right) + (1-p) \cdot \left( \frac{1}{(1-p)^2} - 1 \right) \]

= \[ \frac{1 - p + p^2}{p(1-p)} \]

(b) (10 pts)
According to the problem, the probability that they stop having kid at \( k \)th one should be \( p_{stop} = p^2 \cdot (1-p)^{k-2} + (1-p)^2 \cdot p^{k-2} \).
Thus, the expected number of kids should be

\[
\text{Expected value} = \sum_{i=2}^{\infty} i \cdot (p^2 \cdot (1-p)^{i-2} + (1-p)^2 \cdot p^{i-2})
\]

\[
= \sum_{i=2}^{\infty} i \cdot p^2 \cdot (1-p)^{i-2} + \sum_{i=2}^{\infty} i \cdot (1-p)^2 \cdot p^{i-2}
\]

\[
= \frac{p^2}{1-p} \sum_{i=2}^{\infty} i \cdot (1-p)^{i-1} + \frac{(1-p)^2}{p} \sum_{i=2}^{\infty} i \cdot p^{i-1}
\]

\[
= \frac{p^2}{1-p} \left( \sum_{i=1}^{\infty} i \cdot (1-p)^{i-1} - 1 \right) + \frac{(1-p)^2}{p} \left( \sum_{i=1}^{\infty} i \cdot p^{i-1} - 1 \right)
\]

= applying example 25 here

\[
= \frac{1}{1-p} - \frac{p^2}{1-p} + \frac{1}{p} - \frac{(1-p)^2}{p}
\]

\[
= \frac{1 - p^3 - (1-p)^3}{p(1-p)}
\]

\[
= \frac{3p - 3p^2}{p - p^2} = 3
\]