Solutions

1. (20 pts)

1.a Page 511 #12. (40 pts)

1) (5 pts)

Let $a_n$ be the number of ways to climb $n$ stairs. In order to climb $n$ stairs, a person can start with three ways. The person must either start with a step of one stair and then climb $n - 1$ stairs (and this can be done in $a_{n-1}$ ways) or else start with a step of two stairs and then climb $n - 2$ stairs (and this can be done in $a_{n-2}$ ways) or else start with a step of three stairs and then climb $n - 3$ stairs (and this can be done in $a_{n-3}$ ways). From this analysis we can immediately write down the recurrence relation, valid for all $n \geq 3$: $a_n = a_{n-1} + a_{n-2} + a_{n-3}$.

2) (5 pts)

Since there is one way to climb no stairs (do nothing), clearly only one way to climb one stair, and two ways to climb two stairs (one step twice or two steps at once). The initial conditions are $a_0 = 1, a_1 = 1; a_2 = 2$

3) (5 pts)

$a_8 = a_7 + a_6 + a_5$, and we can calculate $a_3 = 4, a_4 = 7, a_5 = 13, a_6 = 24, a_7 = 44$ with $a_0 = 1, a_1 = 1$ and $a_2 = 2$. Thus, $a_8 = 81$

1.b Page 525 #28. (15 pts)

1) (10 pts)

The associated homogeneous recurrence relation is $a_n = 2a_{n-1}$. We easily solve the annihilator is $E - 2$. The leftover sequence is $< 2(i + 1)^2 >$, which can be annihilators by $(E - 1)^3$. The annihilator is $(E - 2)(E - 1)^3$. So our annihilator rules tell us that $a_n = \alpha 2^n + (\beta + \gamma n + \delta n^2)$. Next we need a particular solution to the given recurrence relation. We can actually look for a function of the form $f(n) = \beta + \gamma n + \delta n^2$ for $a_n = \alpha 2^n + f(n)$. 

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We plug this into our recurrence relation and obtain \( f(n) = \beta + \gamma n + \delta n^2 = 2f(n - 1) + 2n^2 = 2(\beta + \gamma(n - 1) + \delta(n - 1)^2) \). We rewrite this by grouping terms with equal powers of \( n \), obtaining \((\delta 2) n^2 + (4 \delta \gamma)n + (2 \delta + 2 \gamma \beta) = 0\). In order for this equation to be true for all \( n \), we must have \( \delta = 2, 4 \delta = \gamma \), and \( 2 \delta + 2 \gamma \beta = 0 \). This tells us that \( \gamma = 8, \beta = 12, \delta = -2 \). Therefore, the general solution is the sum of the homogeneous solution and this particular solution, \( a_n = \alpha 2^n - 2n^2 - 8n - 12 \).

2) (5 pts)

We plug the initial condition into our solution \( a_1 = 2\alpha - 2 - 8 - 12 = 4, \)
we have \( \alpha = 13 \). The solution is \( a_n = 13 \times 2^n - 2n^2 - 8n - 12 \).

1.c Page 526 #40. (10 pts)

First we reduce this system to a recurrence relation and initial conditions involving only \( an \). If we subtract the two equations, we obtain \( a_n - b_n = 2a_{n-1} \), which gives us \( b_n = a_n - 2a_{n1} \). We plug this back into the first equation to get \( a_n = 3a_{n-1} + 2(a_{n-1} - 2a_{n-2}) = 5a_{n-1} - 4a_{n-2} \), our desired recurrence relation in one variable. Note also that the first of the original equations gives us the necessary second initial condition, namely \( a_1 = 3a_0 + 2b_0 = 7 \). We now solve this problem for \( a_n \) in the usual way. The annihilator is \((E^2 - 5E + 4) = (E - 1)(E - 4)\), and the solution will be \( a_n = a4^n + b \) after solving for the arbitrary constants, is \( a_n = 1 + 2 \times 4^n \). Finally, we plug this back into the equation \( b_n = a_n - 2a_{n-1} \) to find that \( b_n = 1 + 4^n \).

2. (35 pts)

2.a Page 536 #36. (10 pts)

When \( n = 2^k \), let \( f(n_k) = 8f(n_{k-1}) + (n_i)^2 \). The original recurrence gives us the following secondary recurrence for \( n_i, n_{i-1} = n_i/2 \) and \( n_0 = 1 \). The annihilator for this recurrence is \((E - 2)\), the generic solution is \( n_i = \alpha 2^i \). Plugging in the base cases \( n_0 = 1, n_i = 2^i, \) thus \( i = \log(n) \). If we set \( f_i = f(n_i) \), we have \( f_i = 8f_{i-1} + 4^i \). The annihilator is \((E - 4)(E - 8)\), the generic solution is \( \alpha'8^i + \beta'4^i \). Plugging in the base cases \( f_0 = 1, f_1 = 12 \). We have \( 8^i = 8^{\log(n)} = n^3, 4^i = 4^{\log(n)} = n^2 \). Thus, we have \( f(n) = -n^2 + 2n^3 \).

2.b Master Theorem on page 532 (15 pts)

If \( n = b^d \), \( f(n_k) = a f(n_{k-1} + cn_i^d) \), then we have \( f_i = af_{i-1} + (b^d)^i \). The annihilator is \((E - a)(E - b^d)\).

1. If \( a = b^d \), the generic solution is \( f_i = \alpha + \beta(b^d)^i + \gamma i(b^d)^i \), which implies \( f_i = O(i(b^d)^i) \), we have \( f(n) = (\log_b n)n^d \).
2. If $a < b^d$, we have generic solution is $f_i = \alpha a^i + \beta (b^d)^i$, where $f_i = O((b^d)^i) = O(n^d)$.

3. If $a > b^d$, we have $f_i = O(a^i)$, $f_i = b^{\log_b a^i} = (b^i)^{\log_b a} = n^{\log_b a}$

2.c (10 pts)

Let $n = 2^i$, $\sqrt{n} = 2^{i/2},$ and $i = \log_2 n$. $T(n_i) = \sqrt{n_i} \cdot T(n_i-1) + (n_i)$. Making both side divide by $2^k$, we have $T(2^i)/2^i = T(2^{i/2}/2^{i/2}) + 1$. Let $y(i) = T(2^i)/2^i$, we have $y(i) = y(i/2) + 1$. Then we can let $i = 2^k$, $k = \log_2 i$, we have recurrence $y_k = y_{k-1} + 1$. The annihilator for this recurrence is $(E - 1)^2$. Plugging in the base cases $k_0 = 1$, the solution is $y(i) = \log_2 i$. Since we also know that $T(2^i) = 2^i y(i) = n \log_2 \log_2 n$.

3. (25 pts)

3.a (10 pts)

We define $f(10) = 1/(10^2) + 1/(10^3) + 2/(10^4) + 3/(10^5) + 5/(10^6) + 8/(10^7) + \cdots$, let $f(10) - (1/10 \ast f(10))$, we have $9/10 f(10) - 1/(10^2) = 1/(10^4) + 1/(10^5) + 2/(10^6)$, which is equal to $1/100 \ast f(10)$. Thus, we have $9/10 f(10) - 1/100 = 1/100 f(10)$, $f(10) = 1/89$

3.b (15 pts)

Base 8, we have $1/55 = 1/(8^2) + 1/(8^3) + 2/(8^4) + 3/(8^5) + 5/(8^6) + 8/(8^7) + \cdots$, where $55 = 8^2 - 8 - 1$.

To prove it, the idea is similar with the above. We define $f(8) = 1/(8^2) + 1/(8^3) + 2/(8^4) + 3/(8^5) + 5/(8^6) + 8/(8^7) + \cdots$, let $f(8) - (1/8 \ast f(8))$, we have $7/8 f(8) - 1/(8^2) = 1/(8^4) + 1/(8^5) + 2/(8^6)$, which is equal to $1/64 \ast f(8)$. Thus, we have $7/8 f(8) - 1/64 = 1/64 f(8)$, $f(8) = 1/55$