**Illinois Institute of Technology**  
**Department of Computer Science**

**Solutions to Homework Assignment 4**  
**CS 330 Discrete Structures**  
**Spring Semester, 2019**

**Solution:**

1. (a) i. Let $a_n$ be the number of ternary strings of length $n$ that contain two consecutive 0s. If a ternary string of length $n$ ends in 00, then the first $n - 2$ elements can be any value (0, 1 or 2). So there are $3^{n-2}$ such strings. If a ternary string of length $n$ ends in 1 or 2, its initial substring of length $n - 1$ must be a ternary string containing two consecutive 0s. So there are $2a_{n-1}$ such strings. If a ternary string of length $n$ ends in 10 or 20, then its initial substring of length $n - 2$ must be a ternary string containing two consecutive 0s. So there are $2a_{n-2}$ such strings. In total we can find the recurrence relation as:

$$a_n = 2a_{n-1} + 2a_{n-2} + 3^{n-2}$$

ii. Initial conditions are:

$$a_0 = 0, a_1 = 0, a_2 = 1$$

iii. In order to annihilate the sequence:

$$a_n = 2a_{n-1} + 2a_{n-2} + 3^{n-2}$$

the annihilator is $(E^2 - 2E - 2)(E - 3) = (E - \phi)(E - \hat{\phi})(E - 3)$, where $\phi = (1 - \sqrt{3})$ and $\hat{\phi} = (1 + \sqrt{3})$. Thus the generic solution is $a_n = \alpha \phi^n + \beta \hat{\phi}^n + \gamma 3^n$. Given the initial conditions, we can solve the equations and get:

$$a_n = \left(\frac{1}{\sqrt{3}} - \frac{1}{2}\right)(1 - \sqrt{3})^n - \left(\frac{1}{\sqrt{3}} + \frac{1}{2}\right)(1 + \sqrt{3})^n + 3^n$$

Thus we can get $a_6 = 281$.

Or we can solve the recurrence iteratively to find $a_6$:

$$a_0 = 0$$
$$a_1 = 0$$
$$a_2 = 1$$
$$a_3 = 0 + 2 \times 1 + 3^1 = 5$$
$$a_4 = 2 \times 1 + 2 \times 5 + 3^2 = 21$$
$$a_5 = 2 \times 5 + 2 \times 21 + 3^3 = 79$$
$$a_6 = 2 \times 21 + 2 \times 79 + 3^4 = 281$$

(b) i. Find all solutions of the recurrence relation $a_n = -5a_{n-1} - 6a_{n-2} + 42 \cdot 4^n$.

- The homogeneous annihilator is $(E^2 + 5E + 6)(E - 4)$.
- The annihilator factors into $(E + 2)(E + 3)(E - 4)$.
- Thus the generic solution is $a_n = \alpha(-2)^n + \beta(-3)^n + \gamma 4^n$. 


Thus the solutions can be
\[ a_n = \alpha (-2)^n + \beta (-3)^n + \gamma 4^n \]

ii. Find the solution of this recurrence relation with \(a_1 = 56\) and \(a_2 = 278\). If the initial conditions are: \(a_1 = 56\) and \(a_2 = 278\), we have \(a_2 = -5a_1 - 6a_0 + 42 \cdot 4^2\), thus \(a_0 = 19\). So the constraints \(\alpha, \beta, \gamma\) satisfy the equations:

\[
\begin{align*}
\alpha \cdot (-2)^0 + \beta \cdot (-3)^0 + \gamma \cdot 4^0 &= 19 \\
\alpha \cdot (-2)^1 + \beta \cdot (-3)^1 + \gamma \cdot 4^1 &= 56 \\
\alpha \cdot (-2)^2 + \beta \cdot (-3)^2 + \gamma \cdot 4^2 &= 278
\end{align*}
\]

Solving the equations gives us \(\alpha = 1, \beta = 2, \gamma = 16\). So the final solution is
\[ a_n = (-2)^n + 2 \times (-3)^n + 16 \times 4^n \]

(c) Two ways to solve this problem:

i. **Guess and confirm:** we can write out first few values of \(a_n\) and \(b_n\):

\[
\begin{align*}
a_0 &= 1, a_1 = 7, a_2 = 31, a_3 = 127, \ldots \\
b_0 &= 2, b_1 = 5, b_2 = 17, b_3 = 65, \ldots
\end{align*}
\]

It looks like \(a_n = 2^{2n+1} - 1 = 2 \times 4^n - 1\) and \(b_n = 4^n + 1\). Then we check whether it satisfies the recurrence relations:

\[
\begin{align*}
a_n &= 3a_{n-1} + 2b_{n-1} \\
&= 3(2 \times 4^{n-1} - 1) + 2(4^{n-1} + 1) \\
&= 8 \times 4^{n-1} - 1 \\
&= 2 \times 4^n - 1 \\
b_n &= a_{n-1} + 2b_{n-1} \\
&= 2 \times 4^{n-1} - 1 + 2(4^{n-1} + 1) \\
&= 4^n + 1
\end{align*}
\]

Obviously it’s correct. So
\[
\begin{align*}
a_n &= 2 \times 4^n - 1 \\
b_n &= 4^n + 1
\end{align*}
\]

ii. We can also use annihilator to solve this problem. From:

\[
\begin{align*}
a_n &= 3a_{n-1} + 2b_{n-1} \\
b_n &= a_{n-1} + 2b_{n-1}
\end{align*}
\]

we can have:

\[ b_n = a_n - 2a_{n-1} \]

replace it to the first equation we should have

\[
\begin{align*}
a_n &= 3a_{n-1} + 2(a_{n-1} - 2a_{n-2}) \\
&= 5a_{n-1} - 4a_{n-2}
\end{align*}
\]
The annihilator is $E^2 - 5E + 4E$, which can be factored as $(E - 1)(E - 4)$. Thus the solution for $a_n$ is in the form:

$$a_n = \alpha 4^n + \beta$$

Give initial condition $a_0 = 1, a_1 = 7$, we can solve it and get $\alpha = 2, \beta = -1$. Thus

$$a_n = 2 \times 4^n - 1$$

Put it into the first equation, we can solve $b_n$.

$$b_n = 4^n + 1$$

Thus we have:

$$a_n = 2 \times 4^n - 1$$
$$b_n = 4^n + 1$$

2. (a) We have

$$f(1) = 1$$
$$f(n) = 5f(n/4) + 6n$$

Let $n = n_i$ be the $i^{th}$ argument of $f(\cdot)$ from base case $n_0 = 1$. The original recurrence gives us the following secondary recurrence for $n_i$:

$$n_{i-1} = n_i/4 \quad \text{implies} \quad n_i = 4^i$$

Let $t_i = f(n_i) = 5t_{i-1} + 6 \times 4^i$. $t_i$ is annihilated by $(E - 5)(E - 4)$. So the generic solution is

$$t_i = \alpha 5^i + \beta 4^i$$

Plugging the base case $t_0 = 1, t_1 = 29$, we get the exact solution:

$$t_i = 25 \times 5^i - 24 \times 4^i$$

Finally we need to substitute to get a solution for the original recurrence in terms of $n$, by inverting the solution of the secondary recurrence. If $n = n_i = 4^i$, then we have

$$i = \log_4(n)$$

Substituting this into the expression for $t_i$, we get the exact, closed form solution:

$$T(n) = 25 \times 5^\log_4(n) - 24 \times 4^\log_4(n)$$
$$= 25 \times n^{\log_4 5} - 24n$$

(b) Recall the Master Theorem states

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

- If $af\left(\frac{n}{b}\right)/f(n) < 1$, then $T(n) = \Theta(f(n))$
- If $af\left(\frac{n}{b}\right)/f(n) > 1$, then $T(n) = \Theta(n^{\log_b a})$.
- If $af\left(\frac{n}{b}\right)/f(n) = 1$, then $T(n) = \Theta(f(n) \log_b n)$. 

• If none of these three cases apply, you’re on your own.

We begin by setting up the recurrence: \( T(n) = aT\left(\frac{n}{b}\right) + f(n) \), assume \( af\left(\frac{n}{b}\right)/f(n) = K \).

We let \( n = t_i, n/b = t_{i-1} \). Similar to the previous problem, \( t_i = b^i \), and we let

\[
F(i) = T(n) = T(b^i) = aT(b^{i-1}) + f(n) = aF(i-1) + f(n)
\]

For \( f(n) \), we have \( f(n) = \frac{a}{K} f\left(\frac{n}{b}\right) \), and similar to above we let \( G(i) = f(n) = \frac{a}{K} f(b^i-1) = \frac{a}{K} G(i-1) \), where \( n = b^i \) and \( i = \log_b n \). Then, \( G(i) = \left(\frac{a}{K}\right)^i \), and

\[
F(i) = aF(i-1) + \left(\frac{a}{K}\right)^i
\]

whose annihilator is \((E - a)(E - \frac{a}{K})\).

• \( K < 1 \): If \( K < 1, a \neq \frac{a}{K} \), and the generic solution is \( F(i) = c_1a^i + c_2\left(\frac{a}{K}\right)^i \). However, because \( K < 1, \frac{a}{K} > a \), and the growth rate of \( F(i) \) is dominated by \( c_2\left(\frac{a}{K}\right)^i \), and \( F(i) \in O((\frac{a}{K})^i) \), which also means \( T(n) \in O((\frac{a}{K})^i) \).

• \( K > 1 \): Similarly, \( F(i) \)'s growth rate is dominated by \( c_1a^i \) in this case. Then, \( F(i) \in O(a^i) \), which means \( T(n) \in O(a^{\log a n}) = O(n^{\log a}) \).

• \( K = 1 \): Similarly, \( F(i) \)'s annihilator is \((E - a)(E - \frac{a}{K}) = (E - a)^2 \). Then, the generic solution is \( F(i) = (c_1 + c_2)a^i \in O(a^i) = O(n \log n f(n)) \), because \( f(n) = G(i) \in O(a^i) \).

(c)

\[
T(n) = \sqrt{n}T\left(\sqrt{n}\right) + n
\]

We divide \( n \) to both sides of the equation, then we have:

\[
\frac{T(n)}{n} = \frac{T\left(\sqrt{n}\right)}{\sqrt{n}} + 1
\]

Let \( F(n) = \frac{T(n)}{n} \), then the equation becomes:

\[
F(n) = F(\sqrt{n}) + 1
\]

Next, suppose \( n = 2^m \), then \( F(n) = F(2^m) = F(\sqrt{2^m}) = F(2^{\frac{m}{2}}) + 1 \). We again let \( G(m) = F(2^m) \), then we have:

\[
G(m) = G\left(\frac{m}{2}\right) + 1
\]

whose generic solution can be solved as aforementioned, which is \( G(m) = \alpha \log m + \beta \). We undo the substitutions, then we have:

\[
F(2^m) = G(m) = \alpha \log m + \beta
\]

\[
F(n) = F(2^m) = \alpha \log m + \beta = \alpha \log n + \beta
\]

\[
\frac{T(n)}{n} = \alpha \log n + \beta
\]

\[
T(n) = \alpha n \log n + \beta n
\]

Thus

\[
T(n) = \alpha n \log n + \beta n
\]
3. (a) The RHS can be re-written as:

\[ \sum_{i=1}^{\infty} F_i \left( \frac{1}{10} \right)^{i+1} \]

where \( F_i \) is the Fibonacci number. We can write this as \( f(x) = \sum_{i=1}^{\infty} F_i \left( \frac{1}{x} \right)^i \). Then,

\[
\begin{align*}
  f(x) &= \sum_{i=1}^{\infty} F_i \left( \frac{1}{x} \right)^i \\
  &= \frac{1}{x} \sum_{i=1}^{\infty} F_i \left( \frac{1}{x} \right)^i \\
  &= \frac{1}{x} \sum_{i=1}^{\infty} \left[ F_1 \left( \frac{1}{x} \right) + \sum_{i=2}^{\infty} F_i \left( \frac{1}{x} \right)^i \right] \\
  &= \frac{1}{x^2} + \frac{1}{x} \sum_{i=2}^{\infty} F_i \left( \frac{1}{x} \right)^i \\
  &= \frac{1}{x^2} + \frac{1}{x} \sum_{i=2}^{\infty} (F_{i-1} + F_{i-2}) \left( \frac{1}{x} \right)^i \\
  &= \frac{1}{x^2} + \frac{1}{x} \sum_{i=2}^{\infty} F_{i-1} \left( \frac{1}{x} \right)^i + \frac{1}{x} \sum_{i=2}^{\infty} F_{i-2} \left( \frac{1}{x} \right)^i
\end{align*}
\]

Let \( j = i - 1 \) and \( k = i - 2 \). Then,

\[
\begin{align*}
  f(x) &= \frac{1}{x^2} + \frac{1}{x} \sum_{j=1}^{\infty} F_j \left( \frac{1}{x} \right)^{j+1} + \frac{1}{x} \sum_{k=0}^{\infty} F_k \left( \frac{1}{x} \right)^{k+2} \\
  &= \frac{1}{x^2} + \frac{1}{x} \sum_{j=1}^{\infty} F_j \left( \frac{1}{x} \right)^{j+1} + \frac{1}{x^2} \sum_{k=1}^{\infty} F_k \left( \frac{1}{x} \right)^{k+1} \\
  &= \frac{1}{x^2} + \frac{f(x)}{x} + \frac{f(x)}{x^2}
\end{align*}
\]

Therefore, \( f(x) = \frac{1}{x^2 - x - 1} \). Thus we have:

\[
\frac{1}{89} = f(10) = \frac{1}{10^2 - 10 - 1} = 1/10^2 + 1/10^3 + 2/10^4 + 3/10^5 + 5/10^5 + 8/10^7 + \cdots
\]

(b) The above procedure does not change when the base changes. If the base is 8, then 10\(_8\) is equal to 8 when the base is 10. Then, \( f(10\_8) = \frac{10\_8}{10\_8 - 1} = \frac{(10\_8)^2 - 10\_8 - 1}{(10\_8)^2 - 10\_8 - 1} = \frac{10\_8}{10\_8} = 10\_8 \). Therefore, the reciprocal 67 plays the same role if the base becomes 8.

4. **Extra Credit** The puzzle, by Jonah Kagan and edited by Will Shortz, is from the *New York Times* of May 11, 2011. Here is the solution:
All the theme answers relate to the FIBONACCI SERIES (33 across) of Leonardo of PISA (14 across). Patterns found in nature such as the flowering of an ARTICHOKE (17 across), shell of a NAUTILUS (29 across), cochlea in the INNER EAR (42 across), and florets in a SUNFLOWER (58 across) can all be mathematically described the Fibonacci series. The circled squares in the puzzle, which spell GOLDEN RATIO, show a Fibonacci spiral, a pattern created using the Fibonacci series. And, of course, the golden ratio, intimately tied to the Fibonacci series, is usually represented by the Greek letter PHI (33 across), $\phi = \frac{1 + \sqrt{5}}{2} \approx 1.61801$. 

\[ \phi = \frac{1 + \sqrt{5}}{2} \approx 1.61801 \]