

Solutions to Homework Assignment 1

CS 330 Discrete Structures
Fall Semester, 2009

1. Page 19, Problem 28 (b) and (e). Construct a truth table for each compound proposition.

(b) $p \leftrightarrow \neg p$

p	$\neg p$	$p \leftrightarrow \neg p$
F	T	F
T	F	F

(e) $(q \rightarrow \neg p) \leftrightarrow (p \leftrightarrow q)$

p	q	$q \rightarrow \neg p$	$p \leftrightarrow q$	$(q \rightarrow \neg p) \leftrightarrow (p \leftrightarrow q)$
F	F	T	T	T
F	T	T	F	F
T	F	T	F	F
T	T	F	T	F

2. Page 50, problem 60. Express each of the following statements.

$P(x)$ - x is a clear explanation

$Q(x)$ - x is satisfactory

$R(x)$ - x is an excuse

(a) All clear explanations are satisfactory: $\forall x(P(x) \rightarrow Q(x))$

(b) Some excuses are unsatisfactory: $\exists x(R(x) \wedge \neg Q(x))$

(c) Some excuses are not clear explanations: $\exists x(R(x) \wedge \neg P(x))$

(d) (c) does follow from (a) and (b). To show this, take the contrapositive of (a): $\forall x(\neg Q(x) \rightarrow \neg P(x))$ and we can substitute $\neg P(x)$ for $\neg Q(x)$ in (b) and get (c).

3. Page 61, problem 30 (a) and (b). Rewrite each of these statements so that negations only appear within predicates:

(a) $\neg \exists y \exists x P(x, y) \equiv \forall y \forall x \neg P(x, y)$

(b) $\neg \forall x \exists y P(x, y) \equiv \exists x \forall y \neg P(x, y)$

4. Page 149, problem 70 (c). Prove that $\lceil \lceil x/2 \rceil / 2 \rceil = \lceil x/4 \rceil$

Since we are dividing by 4, let $x = 4n + k$ where n is an integer and $0 \leq k < 4$. There are 3 cases.

If $k = 0$ then $\lceil x/4 \rceil = n$, $\lceil x/2 \rceil = 2n$, and $\lceil \lceil x/2 \rceil / 2 \rceil = \lceil n \rceil = n$.

If $0 < k \leq 2$ then $\lceil x/4 \rceil = n + 1$, $\lceil x/2 \rceil = 2n + 1$, and $\lceil \lceil x/2 \rceil / 2 \rceil = \lceil \frac{2n+1}{2} \rceil = \lceil n + \frac{1}{2} \rceil = n + 1$.

If $2 < k \leq 4$ then $\lceil x/4 \rceil = n + 1$, $\lceil x/2 \rceil = 2n + 2$, and $\lceil \lceil x/2 \rceil / 2 \rceil = \lceil \frac{2n+2}{2} \rceil = \lceil n + 1 \rceil = n + 1$.

The statement is true in all three cases.

5. Page 178, problem 20. Describe an algorithm for finding both the largest and the smallest integers in a finite sequence of integers.

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procedure SMALLEST-AND-LARGEST( $a_1, a_2, \dots, a_n$ : integers)
   $min \leftarrow a_1$ 
   $max \leftarrow a_1$ 
  for  $i \leftarrow 2$  to  $n$  do
    if  $a_i < min$  then
       $min \leftarrow a_i$ 
    end if
    if  $a_i > max$  then
       $max \leftarrow a_i$ 
    end if
  end for
end procedure

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6. Page 192, Problem 62. Determine whether $\log n!$ is $\Theta(n \log n)$
 Solve by showing $f(x)$ is $O(g(x))$ and $g(x)$ is $O(f(x))$,
 i.e., $C_1|g(x)| \leq |f(x)| \leq C_2|g(x)|$ when $x > k$
 Big-Oh estimate for $n!$ obtained by noting each term in factorial expansion does not exceed n .

$$n! = 1 * 2 * 3 * \dots * n$$

$$n! \leq n * n * n * \dots * n$$

$$n! \leq n^n \Rightarrow n! \text{ is } O(n^n)$$

$$\log n! \leq \log n^n \Rightarrow \log n! \text{ is } O(n \log n)$$

To obtain Big-Oh estimate for $n \log n$ recognize computational complexity is independent of proportionality constants.

Need to show $n \log n \leq c \log n!$

$$n^n \leq n!^c$$

Let $c = 2$, $n!^2$ is a square of products.

For $n = 3$, $n!^2 = 3! * 3! = 36$ and $n^n = 3^3 = 27$. $n!^2$ will continue to outgrow n^n as $n \rightarrow \infty$

$$2 \log n! \geq n \log n \Rightarrow n \log n \text{ is } O(\log n!)$$

Therefore $\log n!$ is $\Theta(n \log n)$

Note: It's important to be aware that you can't prove $\log n! \in \Theta(n \log n)$ by proving that $n! \in \Theta(n^n)$. We can easily show that exponentials with different bases are in different complexity classes: there is no c and k pair for which $3^n \leq c2^n$ for all $n \geq k$. (Equivalently, $\lim_{n \rightarrow \infty} \frac{3^n}{2^n} = \infty$.) So if you eliminate the log function from this problem before inserting the constant used for big-O proofs, then you will incorrectly conclude that $\log n! \notin \Theta(n \log n)$.

7. Page 280, Problem 30. If H_n is the n 'th harmonic number ($H_n = \sum_{i=1}^n \frac{1}{i}$), prove that $H_1 + H_2 + \dots + H_n = (n+1)H_n - n$

Base case $n = 1$: $H_1 = 1 = 2 \cdot 1 - 1$

Inductive case: To show that if $H_1 + H_2 + \dots + H_n = (n+1)H_n - n$ then $H_1 + H_2 + \dots + H_n + H_{n+1} = (n+2)H_{n+1} - n - 1$.

First we substitute in the inductive hypothesis on the the left side, and then we simplify the right side to the same expression.

$$\begin{aligned}
 H_1 + H_2 + \dots + H_n + H_{n+1} &= (n+1)H_n - n + H_{n+1} \\
 &= (n+1)H_n - n + H_n + \frac{1}{n+1} \\
 &= (n+2)H_n - n + \frac{1}{n+1} \\
 (n+2)H_{n+1} - n - 1 &= (n+2) \left(H_n + \frac{1}{n+1} \right) - (n+1) \\
 &= (n+2)H_n + \frac{n+2}{n+1} - (n+1) \\
 &= (n+2)H_n + 1 + \frac{1}{n+1} - n - 1 \\
 &= (n+2)H_n - n + \frac{1}{n+1}
 \end{aligned}$$

8. Page 281, problem 48: What is wrong with this proof?

“Theorem” For every positive integer n , $\sum_{i=1}^n i = \frac{(n+\frac{1}{2})^2}{2}$

Base case: The formula is true for $n = 1$.

Inductive step: Suppose that $\sum_{i=1}^n i = \frac{(n+\frac{1}{2})^2}{2}$. Then $\sum_{i=1}^{n+1} i = \sum_{i=1}^n i + (n+1)$. by the inductive hypothesis, $\sum_{i=1}^{n+1} i = \frac{(n+\frac{1}{2})^2}{2} + (n+1) = (n^2 + n + \frac{1}{4})/2 + n + 1 = (n^2 + 3n + \frac{9}{4})/2 = (n + \frac{3}{2})^2/2 = [(n+1) + \frac{1}{2}]^2/2$

Answer: The base case was wrong. When $n = 1$, the left hand side is 1, but the right hand side is $(1 + \frac{1}{2})^2/2 = \frac{9}{8}$

9. Bob and Carol are each secretly assigned consecutive positive integers; they each know their own number and that the numbers are consecutive, but they do not know each others number. They are told to sit in a room with a clock that chimes every hour. They cannot communicate in any way, but are told to wait in the room until they can deduce the others number and then leave the room at the next chime of the clock. Prove by induction that the person with the smaller number, n , will leave the room after the n th strike of the clock.

Proof is by induction. We assume that Bob and Carol are both paying attention to what's going on, that they both employ the strategy we use in our proof, and that neither makes mistakes.

Base case: Somebody has the number 1. When the first chime rings, that person will get up and leave the room, because they know there's know smaller number, so the other person must have the number 2.

Inductive case: In a situation where the smallest number was n , the person with that number would have left at the n th strike of the clock (this is the inductive hypothesis). Thus, if the person with number $n + 1$ is sitting in the room at time $n + 1$, and sees his partner has not yet left, he knows the smallest number is greater than n , and therefore that the smallest number is his own $n + 1$, and the larger number is the other persons $n + 2$.