Illinois Institute of Technology
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# Solutions to Homework Assignment 2

CS 330 Discrete Structures
Fall Semester, 2009

1. Page 346, problem 58: Use the product rule to show that there are $2^{2^n}$ different truth tables for propositions in $n$ variables.

   To visualize this problem, suppose $n = 2$. Then we have a truth table as follows.

| $v_1$ | $v_2$ | $f_1(v_1, v_2)$ | $f_2(v_1, v_2)$ | $f_3(v_1, v_2)$ | ... | $f_k(v_1, v_2)$ |
|---|---|---|---|---|---|---|
| F | F | F | T | F | ... | T |
| F | T | F | F | T | ... | T |
| T | F | F | F | F | ... | T |
| T | T | F | F | F | ... | T |

   A truth table consists of $n$ columns for input variables, and **exactly one** output column. This output column represents a single *proposition* in $n$ variables. I have abbreviated $k$ different truth tables here by putting all of the output columns in the same table to make things easier to visualize. Thus, the number of "different truth tables" means the number $k$ of different columns $f_i$ that could appear in this abbreviated truth table.

   Our goal is to explain why $k = 2^{2^n}$

   A truth table with $n$ variables has $2^n$ lines, since each variable can have 2 values. Each line can have a result of T or F, so there are $2^{2^n}$ possible ways to fill the output column of the truth table.

2. Page 362, problem 34: Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have more women than men?

   We will denote the number of women in the committee as $w$. For a majority of the committee to be women, there must be 4,5, or 6 female members (In my solution, we allow the committee to have no men, but I didn't grade you down of you interpreted the problem to require at least one man). For each of these committee sizes, we choose $w$ women from the 15 eligible women, and the remainder are men from the 10 eligible men.

$$\sum_{w=4}^{6} \binom{15}{w}\binom{10}{6-w} = 61425 + 30030 + 5005 = 96460$$

3. Page 369, problem 28; also, prove this by induction

   Show that if $n$ is a positive integer, then

$$\binom{2n}{2} = 2\binom{n}{2} + n^2$$

   (a) using a combinatorial argument

   If I have a set of $2n$ events, then I can partition them into two $n$ event sets, named $A$ and $B$.

   I can select a pair of items in three different ways: I can select both from $A$ (which can be done in $\binom{n}{2}$ ways), I can select both from $B$ (which can be done in $\binom{n}{2}$ ways), or I can match any one of the $n$ items in $A$ with any one of the $n$ items in $B$ (for an additional $n^2$) ways.

(b) by algebraic manipulation

$$\begin{aligned}\binom{2n}{2} &= \frac{2n(2n-1)}{2}\\ &= n(2n-1)\\ &= 2n^2-n\\ &= n^2-n+n^2\\ &= n(n-1)+n^2\\ &= 2\left(\frac{n(n-1)}{2}\right)+n^2\\ &= 2\binom{n}{2}+n^2\end{aligned}$$

(c) by induction

**Base case** $n = 2$

$$\binom{4}{2} = 6$$

$$2\binom{2}{2} + 2^2 = 2(1) + 4 = 6$$

**Inductive case** To show that if $\binom{2n}{2} = 2\binom{n}{2} + n^2$ then $\binom{2(n+1)}{2} = 2\binom{n+1}{2} + (n+1)^2$.

First we will compute a couple of things by a combinatorial argument: (You could also use algebraic manipulation)

If I have a set of $2(n+1)$ events, then I can partition them into set $A$ which contains $2n$ events, and set $B$ which contains 2 events. I can select a pair of items in three different ways: I can select both from $A$ (which can be done in $\binom{2n}{2}$ ways), I can select both from $B$ (which can be done in 1 way), or I can match any one of the $2n$ items in $A$ with any one of the 2 items in $B$ (for an additional $4n$) ways. Thus $\binom{2(n+1)}{2} = \binom{2n}{2} + 1 + 4n$.

If I have a set of $n+1$ events, then I can partition them into set $A$ which contains $n$ events, and set $B$ which contains 1 events. I can select a pair of items in two different ways: I can select both from $A$ (which can be done in $\binom{n}{2}$ ways), or I can match any one of the $n$ items in $A$ with any one of the item in $B$ (for an additional $n$) ways. I cannot select both from $B$. Thus $\binom{n+1}{2} = \binom{n}{2} + n$.

Now we can substitute in the inductive hypothesis:

$$\begin{aligned}\binom{2(n+1)}{2} &= \binom{2n}{2} + 1 + 4n\\ &= 2\binom{n}{2} + n^2 + 1 + 4n\\ &= 2\left(\binom{n}{2} + n\right) + n^2 + 2n + 1\\ &= 2\binom{n+1}{2} + (n+1)^2\end{aligned}$$

4. Page 380, problem 24: How many ways are there to distribute 15 distinguishable objects into five distinguishable boxes so that the boxes have one, two, three, four, and five objects in them respectively?

   We place down labels 1,2,3,4,5 indicating how many balls will go in each box. These are points of reference.

   First, we pick an order for all 15 items. There are 15! possible orders. Then we pick an order for the boxes. There are 5! possible orders. (If you interpreted the problem to conclude that specific boxes needed to have specific numbers of items, and consequently did not include these 5! orders for the boxes, you were not penalized.) In the first box, there will be 1 item. In the second box, there will be 2 items, but their 2! possible orders within the box don't matter. In the third box, there will be 3 items, but their 3! possible orders within the box don't matter. In the fourth box, there will be 4 items, but their 4! possible orders within the box don't matter. In the fifth box, there will be 5 items, but their 5! possible orders within the box don't matter.

   In total, there are

   $$\frac{15! \cdot 5!}{1! \cdot 2! \cdot 3! \cdot 4! \cdot 5!} = 4,540,536,000$$

   combinations.

   Another way of thinking about this problem:

   We select 1 items that will go at label 1. This can be done in $\binom{15}{1}$ ways. From the remaining 14 items, we select 2 items that will go at label 2. This can be done in $\binom{14}{2}$ ways. From the remaining 12 items, we select 3 items that will go at label 3. This can be done in $\binom{12}{3}$ ways. From the remaining 9 items, we select 4 items that will go at label 5. This can be done in $\binom{9}{4}$ ways. From the remaining 5 items, we select 5 items that will go at label 5. This can be done in $\binom{5}{5}$ ways. This gives a total of $\binom{15}{1}\binom{14}{2}\binom{12}{3}\binom{9}{4}\binom{5}{5}$ ways to group the items.

   These groups can be placed in the boxes in any order. This means there are an additional 5! ways to order these groups.

   When we combine these events, we use the product rule, so the total answer is $\binom{15}{1}\binom{14}{2}\binom{12}{3}\binom{9}{4}\binom{5}{5}5!$ (If you expand the binomial coefficients and cancel out the matching factorials, you will see that these two solutions are algebraically equivalent.)

5. Page 381, problem 58. How many ways are there to distribute five balls into seven boxes if each box must have at most one ball in it and

   (a) both the balls and the boxes are labeled? This is a permutation. Hold the balls in a fixed order, and select 5 boxes whose order matters relative to the balls.

   $$_7P_5 = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2520$$

   (d) both the balls and the boxes are unlabeled?

   There is only one way to do this, which is to put 5 balls in 5 different boxes (order doesn't matter) and leave two boxes empty.

6. Page 389, problem 26: Corollary 2 in Section 5.4 stated that if $n$ is a positive integer, then

   $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

Give a combinatorial proof of this corollary, by setting up a correspondence between the subsets of a set with an even number of elements, and the subsets of this set with an odd number of elements. [Hint: Take an element $a$ in the set. Set up the correspondence by putting $a$ in the subset if it is not already in it and taking it out if it is in the subset.]

This corollary is equivalent to saying that the sum of the numbers $\binom{n}{k}$ for even $k$ is equal to the sum of the numbers $\binom{n}{k}$ for odd $k$.

Since $\binom{n}{k}$ is counts the number of subsets of size $k$ in a set with $n$ elements, we need to show that a set has as many even-sized subsets as it has odd-sized subsets.

Let's call our $n$-item set $A$. Remove item $a$ from it, and call the remainder $B$. For every subset $C$ of $B$, there are two corresponding subsets of $A$: $C$ and $C \cup \{a\}$. If $C$ has an odd number of items, then $C \cup \{a\}$ has an even number, and vice versa. By covering all subsets of $B$, we cover all subsets of $A$, and so the number of even subsets of $A$ is the same as the number of odd subsets of $A$.