Illinois Institute of Technology
Department of Computer Science

Solutions to Homework Assignment 4

CS 330 Discrete Structures
Fall Semester, 2009

1. Page 458, (Sec 7.1) exercise 28, solve using the operator method discussed in class

   (a) Find a recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take one, two, or three stairs at the same time.

   Let a_n be the number of ways to climb n stairs. A person can start with a step of one step and climb the remaining stairs in a_{n-1} ways. He start with a step of two steps and climb the remaining stairs in a_{n-2} ways. He start with a step of three steps and climb the remaining stairs in a_{n-3} ways.

$$a_n = a_{n-1} + a_{n-2} + a_{n-3}$$

   (b) What are the initial conditions?

$$a_0 = 1, a_1 = 1, a_2 = 2$$

   (c) How many ways can this person climb a flight of eight stairs?

$$\begin{aligned} E^3 \langle a_n \rangle &= E^2 \langle a_n \rangle + E \langle a_n \rangle + \langle a_n \rangle \\ (E^3 - E^2 - E - 1) \langle a_n \rangle &= \langle 0 \rangle \end{aligned}$$

   I don't know how to factor this, so I asked Maxima to do this for me.

$$a_n = \left(\left(3^{-\frac{3}{2}}\sqrt{11} + \frac{19}{27}\right)^{\frac{1}{3}} + \frac{4}{9\left(3^{-\frac{3}{2}}\sqrt{11} + \frac{19}{27}\right)^{\frac{1}{3}}} + \frac{1}{3} \right)^n \alpha +$$
$$\left(\left(3^{-\frac{3}{2}}\sqrt{11} + \frac{19}{27}\right)^{\frac{1}{3}} \left(\frac{\sqrt{3}\,i}{2} - \frac{1}{2}\right) + \frac{4\left(-\frac{\sqrt{3}\,i}{2} - \frac{1}{2}\right)}{9\left(3^{-\frac{3}{2}}\sqrt{11} + \frac{19}{27}\right)^{\frac{1}{3}}} + \frac{1}{3} \right)^n \beta +$$
$$\left(\frac{4\left(\frac{\sqrt{3}\,i}{2} - \frac{1}{2}\right)}{9\left(3^{-\frac{3}{2}}\sqrt{11} + \frac{19}{27}\right)^{\frac{1}{3}}} + \left(3^{-\frac{3}{2}}\sqrt{11} + \frac{19}{27}\right)^{\frac{1}{3}} \left(-\frac{\sqrt{3}\,i}{2} - \frac{1}{2}\right) + \frac{1}{3} \right)^n \gamma$$

   Numerically, that's $a_n = \alpha(1.839286758257819)^n + \beta(-0.6062907292072i - 0.41964337760708)^n + \gamma(0.6062907292072i - 0.41964337760708)^n$

   I'm not going to waste more paper by setting up the equations, but I asked Maxima to solve the system of equations for finding the constants α, β, γ given the initial conditions. The results:

$$\alpha = \frac{\sqrt{3}\left(36\left(\frac{\sqrt{11}}{3\sqrt{3}}+\frac{19}{27}\right)^{\frac{2}{3}}+12\left(\frac{\sqrt{11}}{3\sqrt{3}}+\frac{19}{27}\right)^{\frac{1}{3}}+19\right)+9\sqrt{11}}{\sqrt{3}\left(81\left(\frac{\sqrt{11}}{3\sqrt{3}}+\frac{19}{27}\right)^{\frac{4}{3}}+36\left(\frac{\sqrt{11}}{3\sqrt{3}}+\frac{19}{27}\right)^{\frac{2}{3}}+16\right)}$$

The other two are longer and I can't fit them on the page.

Numerically, they are:

$$\alpha = 0.51903164996324$$
$$\beta = 3.3337193452310902\,10^{-4}\,i\,(778.4573268078414\,i + 426.6177683774731)$$
$$\gamma = \frac{60038.74489044691 - 51557.55775384155\,i}{250284.7595497698\,i - 94185.57816942238}$$

Clearly we're not going to determine a_8 this way. (Though I'm looking for you to get the correct annihilator when you solve this problem — you can't give up before writing that.)

So the solution that the textbook proposes for computing a_8 is just to do it iteratively, and that's what we'll use to solve this problem: $a_0 = 1$, $a_1 = 1$, $a_2 = 2$, $a_3 = 1+1+2 = 4$, $a_4 = 1+2+4 = 7$, $a_5 = 2+4+7 = 13$, $a_6 = 4+7+13 = 24$, $a_7 = 7+13+24 = 44$, and lastly $a_8 = 13+24+44 = 81$.

2. Page 458, (Sec 7.1) exercise 30, solve using the operator method discussed in class

   A string that contains only 0s, 1s, and 2s is called a ternary string.

   (a) Find a recurrence relation for the number for the number of ternary strings that contain two consecutive 0s.

   We can start with either a 1 or a 2, and continue with a string containing two consecutive 0's, we could start with a 01 or 02 and continue with a string containing two consecutive 0's, or we can start with two consecutive 0's and continue with any ternary string of length $n-2$.

$$a_n = 2a_{n-1} + 2a_{n-2} + 3^{n-2}$$

   (b) What are the initial conditions?

   $a_0 = a_1 = 0$ because with 0 or 1 character only, you can't have two consecutive 0s.

   (c) Who many ternary strings of length six contain two consecutive 0s?

$$\begin{aligned}
E^2\langle a_n\rangle &= 2E\langle a_n\rangle + 2\langle a_n\rangle + \langle 3^n\rangle \\
(E^2-2E-2)\langle a_n\rangle &= \langle 3^n\rangle \\
(E^2-2E-2)(E-3)\langle a_n\rangle &= \langle 0\rangle \\
(E-(1+\sqrt{3}))(E-(1-\sqrt{3}))(E-3)\langle a_n\rangle &= \langle 0\rangle \\
a_n &= \alpha(1+\sqrt{3})^n + \beta(1-\sqrt{3})^n + \gamma 3^n
\end{aligned}$$

   Find one more initial condition: $a_2 = 2a_1 + 2a_0 + 3^0 = 1$

Solve for the initial conditions:

$$\begin{aligned} 0 &= \alpha + \beta + \gamma \\ 0 &= \alpha(1+\sqrt{3}) + \beta(1-\sqrt{3}) + \gamma 3 \\ 1 &= \alpha(1+\sqrt{3})^2 + \beta(1-\sqrt{3})^2 + \gamma 3^2 \end{aligned}$$

Using Maxima to solve linear system tells me that $\alpha = -\frac{1}{4\sqrt{3}-6} = -1.077350269189626, \beta = \frac{\sqrt{3}-2}{2\sqrt{3}} = 0.077350269189626, \gamma = 1$
So $a_n = -\frac{1}{4\sqrt{3}-6} \cdot (1+\sqrt{3})^n + \frac{\sqrt{3}-2}{2\sqrt{3}} \cdot (1-\sqrt{3})^n + 3^n$.
Computing $a_6 = 281$, and you can verify this by computing the recurrence iteratively.

3. Page 472, (Sec 7.2) exercise 32. Find the solution of the recurrence relation $a_n = 2a_{n-1} + 3 \cdot 2^n$, using the operator method discussed in class

$$\begin{aligned} E\langle a_n\rangle &= 2\langle a_n\rangle + 3 \cdot 2^n \\ (E-2)\langle a_n\rangle &= 3 \cdot 2^n \\ (E-2)^2\langle a_n\rangle &= \langle 0\rangle \\ \langle a_n\rangle &= \langle(\alpha + \beta n)2^n\rangle \end{aligned}$$

4. Solve the simultaneous recurrence relations

$$\begin{aligned} a_n &= a_{n-1}/2 + 2b_{n-1} \\ b_n &= 2a_{n-1} + b_{n-1}/2 \end{aligned}$$

with initial values $a_0 = b_0 = 1$
Rearrange the second recurrence to $a_{n-1} = b_n/2 - b_{n-1}/4$
Substitute into the first recurrence: $b_{n+1}/2 - b_n/4 = b_n/4 - b_{n-1}/8 + 2b_{n-1}$
Consolidate these terms:

$$\begin{aligned} b_{n+1}/2 - b_n/2 - (15/8)b_{n-1} &= 0 \\ (.5E^2 - .5E - \frac{15}{8})\langle b_n\rangle &= \langle 0\rangle \\ (E+\frac{3}{2})(E-\frac{5}{2})\langle b_n\rangle &= \langle 0\rangle \\ \langle b_n\rangle &= \left\langle \alpha\left(\frac{-3}{2}\right)^n + \beta\left(\frac{5}{2}\right)^n \right\rangle \end{aligned}$$

Substitute back into the (rearranged second equation) to find $\langle a_n\rangle$:

$$\begin{aligned}
a_{n-1} &= b_n/2 - b_{n-1}/4 \\
&= \frac{\alpha}{2}\left(\frac{-3}{2}\right)^n + \frac{\beta}{2}\left(\frac{5}{2}\right)^n - \frac{\alpha}{4}\left(\frac{-3}{2}\right)^{n-1} + \frac{\beta}{4}\left(\frac{5}{2}\right)^{n-1} \\
&= -\alpha\left(\frac{-3}{2}\right)^{n-1} + \frac{3\beta}{2}\left(\frac{5}{2}\right)^{n-1} \\
a_n &= -\alpha\left(\frac{-3}{2}\right)^{n} + \frac{3\beta}{2}\left(\frac{5}{2}\right)^{n}
\end{aligned}$$

Substitute the initial conditions, and solve for α and β:

$$\begin{aligned}
1 &= -\alpha + \frac{3\beta}{2} \\
1 &= \alpha + \beta
\end{aligned}$$

We get $\alpha = 1/5$ and $\beta = 4/5$

Hence,

$$\begin{aligned}
a_n &= -0.2\left(\frac{-3}{2}\right)^n + 1.2\left(\frac{5}{2}\right)^n \\
b_n &= 0.2\left(\frac{-3}{2}\right)^n + 0.8\left(\frac{5}{2}\right)^n
\end{aligned}$$

5. Page 484, exercise 34, solve using secondary recurrences together with the operator method discussed in class and given in the notes.

   Find $f(n)$ when $n = 4^k$, where f satisfies the recurrence relation $f(n) = 5f\left(\frac{n}{4}\right) + 6n$ with $f(1) = 1$.

   Let $n = u_m$, so $f(u_m) = 5f\left(\frac{u_m}{4}\right) + 6u_m = 5f(u_{m-1}) + 6u_m$.

   This means $u_m = 4u_{m-1}$, so we can easily see that $n = u_m = 4^m$. (The annihilator is $(E-4)$)

   Now substitute $f(n)$ with f'_m. So $f'_m = 5f'_{m-1} + 6 \cdot 4^m$.

$$\begin{aligned}
f'_m &= 5f'_{m-1} + 6\cdot 4^m \\
E\langle f'_m\rangle &= 5\langle f'_m\rangle + 24\cdot\langle 4^m\rangle \\
(E-5)\langle f'_m\rangle &= 24\cdot\langle 4^m\rangle \\
(E-5)(E-4)\langle f'_m\rangle &= \langle 0\rangle \\
\langle f'_m\rangle &= \langle \alpha 5^m + \beta 4^m\rangle \\
f(u_m) &= \alpha 5^m + \beta 4^m \\
f(n) &= \alpha 5^{\log_4 n} + \beta n
\end{aligned}$$

We need another initial condition: $f(4) = 5f(1) + 6 * 4 \cdot 2 = 29$

$$\begin{aligned} 1 &= \alpha + \beta \\ 29 &= 5\alpha + 4\beta \\ \alpha &= 25 \\ \beta &= -24 \end{aligned}$$

6. Derive all three cases of the Master Theorem on page 479 (we did parts of this in class)

   Recall the Master Theorem states

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

   - if $af\left(\frac{n}{b}\right) = Kf(n)$ for some $K < 1$ then $T(n) \in \Theta(f(n))$
   - if $af\left(\frac{n}{b}\right) = Kf(n)$ for some $K > 1$ then $T(n) \in \Theta(n^{\log_b a})$
   - if $af\left(\frac{n}{b}\right) = f(n)$ then $T(n) \in \Theta(f(n) \log_b n)$
   - if none of these three cases apply, then you're on your own.

   We begin by setting up the recurrence: $T(n) = aT\left(\frac{n}{b}\right) + f(n)$

   We will replace n with a linear recurrence $T(u_m) = aT\left(\frac{u_m}{b}\right) + f(n) = aT(u_{m-1}) + f(n)$, so $u_m = bu_{m-1}$. This recurrence is easy to solve: it's $n = u_m = b^m$. We'll also note the inverse: $m = \log_b n$

   From the three cases in the Master Theorem, we also know that $\frac{a}{K}f\left(\frac{n}{b}\right) = f(n)$. If we rewrite this in terms of u_m, $\frac{a}{K}f(u_{m-1}) = f(u_m)$, and rewrite it further still as $\frac{a}{K}f'_{m-1} = f'_m$, we can see that $f'_m = \left(\frac{a}{K}\right)^m$.

   We are now ready to rewrite our man recurrence $T(n)$ to t_n. By substituting our above derivations, $t_m = at_{m-1} + f'_m$. Since f'_m is already solved, this is equal to $t_m = at_{m-1} + \left(\frac{a}{K}\right)^m$

   Now we can solve the recurrence. There are two cases here.

   The first case is when $K \neq 1$, and it corresponds to the first *two* cases in the Master Theorem. The annihilator for t_m is $(E - a)\left(E - \frac{a}{K}\right)$, so $t_m = \alpha a^m + \beta\left(\frac{a}{K}\right)^m = \alpha a^m + \beta f'_m$. Undoing all of our substitutions, $T(n) = \alpha a^{\log_b n} + \beta f(n)$. And the exponent can be rearranged making $T(n) = \alpha n^{\log_b a} + \beta f(n)$.

   If $K > 1$, then $f(n)$ grows more slowly than $n^{\log_b a}$. So $T(n) \in \Theta(n^{\log_b a})$ If $K < 1$, then $f(n)$ grows more quickly than $n^{\log_b a}$. So $T(n) \in \Theta(f(n))$

   In the second case, when $K = 1$, the annihilator $(E - a)\left(E - \frac{a}{K}\right) = (E - a)^2$, so $t_m = (\alpha + \beta m)a^m = (\alpha + \beta m)f'_m$. Undoing all of our substitutions, this tells us that $T(n) = (\alpha + \beta \log_b n)f(n) \in \Theta(f(n) \log_b n)$

7. In the notes, the recurrence $T(n) = \sqrt{n} \cdot T(\sqrt{n}) + n$ is solved by a recursion tree, and again by the "guess-and-confirm" method in section 1.6.3. Solve it by means of a secondary recurrence together with the operator method.

$$T(n) = \sqrt{n}T(\sqrt{n}) + n$$

   Divide the whole equation by n.

$$\frac{T(n)}{n} = \frac{T(\sqrt{n})}{\sqrt{n}} + 1$$

Let's substitute $n = 2^m$ and $\frac{T(n)}{n} = \frac{T(2^m)}{2^m} = g(m)$. We note that $m = \lg n$.

$$g(m) = g\left(\frac{m}{2}\right) + 1$$

Now, we can use a secondary recurrence, by letting $m = u_k$

$$g(u_k) = g\left(\frac{u_k}{2}\right) + 1 = g\left(u_{k-1}\right) + 1$$

This means $u_k = 2u_{k-1}$ so we solve u_k and find $m = u_k = 2^k$ and therefore $k = \lg m$.
Now, let $g'_k = g(u_k)$.

$$g'_k = g'_{k-1} + 1$$

We can see that the annihilator for this equation is $(E-1)^2$, so $g_k = \alpha + \beta k$.
Now, all that's left is to undo all of the substitutions and the division that we started with:

$$\begin{aligned} g_k &= \alpha + \beta k \\ g(u_k) = g(m) &= \alpha + \beta \lg m \\ \frac{T(n)}{n} &= \alpha + \beta \lg \lg n \\ T(n) &= \alpha n + \beta n \lg \lg n \end{aligned}$$