1. This program computes quotients and remainders.

\[
\begin{align*}
  r &:= a \\
  q &:= 0 \\
  &\textbf{while } r \geq d \\
  &\textbf{begin} \\
  &\quad r := r - d \\
  &\quad q := q + 1 \\
  &\textbf{end}
\end{align*}
\]

Verify that it is partially correct with respect to the initial assertion “\(a\) and \(d\) are positive integers” and the final assertion “\(q\) and \(r\) are integers such that \(a = dq + r\) and \(0 \leq r < d\).”

2. As in the notes, (extended) binary trees are directed graphs with “nodes” and “links”, and are defined recursively as follows:

(a) there are two kinds of nodes: NULL and non-NULL. Non-NULL nodes have two links, called left-child and right-child (they can also have other information, which we ignore).

(b) one NULL node is a binary tree

(c) If we have two disjoint binary tree \(T_1\) and \(T_2\), then a binary tree (which we call \(T\)) can be constructed with a root non-NULL node, whose left and right children are \(T_1\) and \(T_2\) respectively.

Prove that for any binary tree, the number of NULL nodes is one more than the number of non-NULL nodes.

3. Characterize each of the following recurrence equations using the master method (assuming that \(T(n) = c\) for \(n \leq d\), for constants \(c > 0\) and \(d \geq 1\)).

a. \(T(n) = 2T(n/2) + \sqrt{n}\)

b. \(T(n) = 8T(n/2) + n^2\)

c. \(T(n) = 16T(n/2) + n^4\)

d. \(T(n) = 7T(n/3) + n\)

e. \(T(n) = 9T(n/3) + n^{3.1}\)

You may use without proof:
Theorem 1 (Master Theorem) Let $f$ be an increasing function that satisfies the recurrence relation:

$$f(n) = af(n/b) + cn^d$$

whenever $n = b^k$, where $k$ is a positive integer, $a \geq 1$, $b$ is an integer greater than 1, and $c$ and $d$ are real numbers with $c$ positive and $d$ nonnegative. Then:

$$f(n) \begin{cases} 
O(n^d) & \text{if } a < b^d \\
O(n^d \log n) & \text{if } a = b^d \\
O(n^\log_b a) & \text{if } a > b^d. 
\end{cases}$$

Good luck!