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1 Functions

A function $f$ from a domain $A$ to a codomain $B$ is denoted as $f : A \to B$, and it describes assignments of each element from $A$ to exactly one element of $B$. We write $f(a) = b$ for an assignment from $a$ to $b$, and this is called as a mapping as well. The most important property is that an element in the domain can be mapped to one and only one element in the codomain. Notably, every element in the domain must be mapped to one element in the domain in order to define the mapping as a function, but elements in the codomain may remain unmapped.

This is purely a mathematical definition, and it can be specified in many different ways. Some examples include:

- An explicit statement: Students in CS330 and their grades.
- A formula: $f(x) = x^2 + 2x$
- A program: A java program that computes $n!$ given $n$ as the input.

* Can you identify the domain and codomain of the above examples? +1

1.1 Types of functions

Depending on how elements in the domain are mapped to the elements in the codomain, we have injective functions, surjective functions, and bijective functions.

- Injection: Also called one-to-one functions. In such functions, all elements in the domain are mapped to distinct elements in the codomain.
  * Recall the quantifiers. Can you express this with a predicate and nested quantifiers? +5
- Surjection: Also called onto functions. In such functions, every element in the codomain is mapped by at least one element in the domain.
- Bijection: Also called one-to-one correspondent functions. Functions that are both injective and surjective are called bijective.

1.2 Inverse functions

If a function $f : A \to B$ is bijective, we can define inverse function $f^{-1} : B \to A$, where $f(x) = y \iff f^{-1}(y) = x$.

* Can you argue why inverse functions of non-bijective functions cannot be defined? +2

1.3 Composition of functions

Different functions can be chained. For example, for two functions $f : A \to B, g : B \to C$, a chained function $g \cdot f$ refers to a function $g \cdot f : A \to C$. The reason we use $g \cdot f$ instead of $f \cdot g$ to denote this chained function is because given an $x \in A$, $g \cdot f(x) = g(f(x))$. 
2 Sets

Sets are one of the most important structures in the discrete mathematics (and computer science). A set is an unordered collection of elements (also called as members). It is a convention that elements are denoted with non-capital letters and sets are denoted with capital letters.

- If an element $a$ belongs to a set $A$, we write $a \in A$. Otherwise, we write $a \notin A$.

2.1 Important sets

Some sets of numbers are listed below.

- $N$: Natural numbers. The definition of natural numbers may be different in different contexts. In the Number Theory, natural numbers refer to positive integers, and in the Set Theory, they refer to non-negative integers including 0.
- $Z$: Integers.
- $Z^+$: Positive integers.
- $R$: Real numbers.
- $R^+$: Positive real numbers.
- $C$: Complex numbers.
- $Q$: Rational numbers.

2.2 Describing a set

A generic way to describe a finite set is listing all elements in \{···\}. However, this is not possible for infinite sets. One of the most frequently used ways to describe an infinite set is

$$S = \{x|\text{description of } x \text{ which belongs to } S\}$$

For example, the set of real numbers between 0 and 10 (non-inclusive) can be

$$S = \{x \in R|0 < x < 10\} \text{ or } S = \{x|x \in R \land 0 < x < 10\} \text{ or } S = \{x|x \text{ is a real number between 0 and 10}\}$$

All of them refer to the same set.

2.3 Universe set and empty set

- Universe set: The set containing everything that is currently under consideration. Often denoted by $U$.
  - In the context of the integers, $Z$ is the universe set.
- Empty set: The set with no elements. Except extreme cases, denoted by $\emptyset$.
  - $\{\emptyset\}$ is different from $\emptyset$. 
2.4 Set of sets

Anything can be elements in a set (even another sets!). For example, \{\{1, 2\}, \{2, 3\}\} is a valid set, and \{\{1, 2\}, 3\} is a valid set too. This explains why \(\emptyset \neq \emptyset\).

2.5 Subset and superset

A set \(A\) is a subset of \(B\) if and only if every element of \(A\) also belongs to \(B\), and this relationship is denoted as \(A \subseteq B\). In this case, \(B\) is called as a superset of \(A\). The predicate version of this statement is:

\[
\forall x (x \in A \rightarrow x \in B)
\]

* An empty set is a subset of any set. Can you argue this statement based on the above predicate? +2

Specially, if a set \(A\) is a subset of \(B\) but it is known that \(A \neq B\), \(A\) is a proper subset of \(B\), and we write \(A \subset B\).

2.6 Set cardinality

Cardinality of a set \(A\) refers to the number of distinct elements in the set, and the cardinality of a set \(A\) is denoted as \(|A|\). For example

- \(A = \{1, 3, 5, 7\} \Rightarrow |A| = 4\)

One significant principle about the set cardinality is the following one.

**Principle of inclusion and exclusion** For any two sets \(S_1, S_2\), we have:

\[
|S_1 \cup S_2| = |S_1| + |S_2| - |S_1 \cap S_2|
\]

and for any three sets \(S_1, S_2, S_3\), we have:

\[
|S_1 \cup S_2 \cup S_3| = |S_1| + |S_2| + |S_3| - |S_1 \cap S_2| - |S_1 \cap S_3| - |S_2 \cap S_3| + |S_1 \cap S_2 \cap S_3|
\]

We will come back to discuss the more general form of the principle of inclusion and exclusion after discussing the permutation and combination.

2.7 Power set

A powerset of a set \(A\) is the set of all subsets of \(A\), often denoted as \(2^A\). For example

- \(A = \{1, 3, 5\} \Rightarrow 2^A = \{\emptyset, \{1\}, \{3\}, \{5\}, \{1, 3\}, \{1, 5\}, \{3, 5\}, \{1, 3, 5\}\}\)

Notably, the cardinality of \(2^A\) is always equal to \(2^{|A|}\). In the above example, it can be verified that \(|A| = 3\) and \(|2^A| = 2^{|A|} = 2^3 = 8\). Later when we discuss rule of sum and rule product, we will discuss what is the theory behind.
2.8 Cartesian product

A cartesian product of two sets $A$ and $B$ is denoted by $A \times B$, and it is the set of ordered pairs $(a, b)$ as below:

$$A \times B = \{(a, b) | a \in A \land b \in B\}$$

For example

- $A = \{a, b\} \land B = \{1, 2\} \Rightarrow A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$
- $A = \{a, b\} \land B = \{b, c\} \Rightarrow A \times B = \{(a, b), (a, c), (b, b), (b, c)\}$
- $A = \{a, b\} \land B = \{a, b\} \Rightarrow A \times B = \{(a, a), (a, b), (b, a), (b, b)\}$

Here, $(a, b) \neq (b, a)$ because they are ordered pairs.

2.9 Set operations

I’m omitting union and intersection and presenting difference only.

$$A - B = \{x | x \in A \land x \notin B\}$$

3 Rule of sum and rule of product

Rule of sum and rule of produce are two elementary techniques for counting the number of configurations satisfying certain constraints. Such techniques are fundamental to the probability theory and many other branches of Discrete Mathematics.

Often, we are asked to count the number of different ways that certain events may happen, and the solution space may be so large that we cannot enumerate all the events one by one. In such cases, we often break down the events into simpler ones whose configurations are easy to count. For example, when we try to calculate the number of ways that the number of heads is equal to 5 in 10 series of coin toss, we consider each toss as an event and apply the rule of product and rule of sum to combine and merge the individual events to result in more complicated events.

3.1 Definitions

Rule of Sum  Suppose an event $E$ occurs when $E_1$ or another separate $E_2$ occurs, meaning that $E_1$ and $E_2$ can’t occur at the same time, and suppose the number of different ways $E_1$ will occur is $e_1$ and $E_2$ will occur is $e_2$ respectively. Then, the number of different ways $E$ will occur is $e_1 + e_2$. This is called rule of sum, and it generalizes to $E$ with multiple events $\{E_1, E_2, \cdots\}$ instead of only two as well.

- Such situation is observed when $E_1$ and $E_2$ describe different cases of $E$, and it is either $E_1$ or $E_2$ that occurs (but not both at the same time).

For example,

- $E : 2$ out of 3 coins are head when 3 coins are tossed.
  - $E_1 : 1$st and 2nd coins are head and 3rd one is tail.
– $E_2$: 1st and 3rd coins are head and 2nd one is tail.
– $E_3$: 2nd and 3rd coins are head and 1st one is tail.

\[ |E| = |E_1| + |E_2| + |E_3| = 1 + 1 + 1 = 3. \]

**Rule of Product** Suppose an event $E$ occurs when $E_1$ and another separate $E_2$ occurs, meaning that $E_1$ and $E_2$ can’t occur at the same time, and suppose the number of different ways $E_1$ will occur is $e_1$ and $E_2$ will occur is $e_2$ respectively. Then, the number of different ways $E$ will occur is $e_1 e_2$. This is called rule of product, and it generalizes to $E$ with multiple events \{ $E_1, E_2, \cdots$ \} instead of only two as well.

• Such situation is observed when $E_1$ and $E_2$ describe different steps of $E$, and both $E_1$ and $E_2$ must occur in order to have $E$.

For example,

• $E$: The first two coins are head when 3 coins are tossed in series (one by one).
  – $E_1$: For the first coin, the number of ways to satisfy $E$ is 1 since it has to be head.
  – $E_2$: For the second coin, the number of ways to satisfy $E$ is 1 since it has to be head.
  – $E_3$: For the third coin, the number of ways to satisfy $E$ is 2 since it can be any side.

\[ |E| = |E_1| \cdot |E_2| \cdot |E_3| = 2. \]

The word *separate* in these two rules is very important, and it will be precisely defined later.

### 3.2 Three simple examples

• Suppose there are 5 short-sleeve shirts, 4 long-sleeve shirts, 4 pairs of pants, 7 ties, and 1 pair of shoes. How many ways can someone choose a set of outfit each morning? For a shirt one can choose either a long- or a short-sleeve shirts ($5 + 4 = 9$ possible shirts), a pair of pants (4 possible pairs of pants), to wear a tie or not ($7$ possible ties or no tie at all, resulting in $8$ choices in all), and only 1 pair of shoes. The total number of ensembles is thus $(5 + 4) \times 4 \times (7 + 1) \times 1 = 288$.

• In a deck of cards we have 4 suits and 13 values (2 through ace), giving a total of $4 \times 13 = 52$ cards.

• A zip code is made of 5 decimal digits, so the number of possible zip codes is $10 \times 10 \times 10 \times 10 = 10^5 = 100000$. The number of zip codes with only odd digits is $5 \times 5 \times 5 \times 5 \times 5 = 10^5 = 3215$. The number of zip codes with no repeated digit is $10 \times 9 \times 8 \times 7 \times 6 = 30240$.

### 3.3 Bigger examples

* This example is entirely from Professor Reingold’s lecture note for CS330 Spring 2015.

#### 3.3.1 Rule of sum

**Example 1** In the figure below, any path of neighboring diamonds from the top “A” to the bottom “A” spells out the word “abracadabra”; furthermore, no other type of path will spell out this word. Then, how many ways are there to spell out the sequence of letters A-B-R-A-C-A-D-A-B-R-A?
For this example, an “event” is “arriving at a particular cell of the diamond from the top cell only by steps that are down-and-right or down-and-left.” As the following diagram depicts, to reach an event $E$ (labelled by the cell “E”) we must either go through event $E_1$ or event $E_2$:

Suppose we can arrive at the cell labeled $E_1$ (this is the event $E_1$) in $e_1$ ways and at the cell $E_2$ (this is the event $E_2$) in $e_2$ ways. Then we can apply the rule of sums to see that event $E$ can happen in exactly $e_1 + e_2$ ways. Note that there is only one way to arrive at a cell along the top left or top right boundaries. We can now compute the number of ways to spell “abracadabra” as follows:

- Fill in the $6 \times 6$ grid with ones along the top left and right boundaries
- apply the rule of sum by adding the two numbers in the cells above an empty cell
- write the resulting sum in the empty cell.

Our hard work pays off with the gleeful conclusion that “abracadabra” can be spelled out in 252 different ways. You might recognize the numbers in our diagram as the entries in Pascal’s triangle.

**Example 2** For our second example, we choose an instance where the rule of sum is *not* applicable. Suppose that of the roughly 200 students in CS 330, 150 are taking Math 247, and 100 are taking Physics 113. How many of the CS 330 students have taken *either* Math 247 *or* Physics 113? Applying the rule of sum suggests 250 out of the 200 students in CS 330 have taken one course or the other! This nonsense results from applying the rule of sum to events that are *not* separate from one another—there are enterprising souls who are enrolled in both Math 247 and Physics 113, as well as some students who are currently taking neither. Thus, the correct answer depends on how many of the students took both courses, not just either course. This tells us how to make the notion of “separate” events precise:

Events $E_1$ and $E_2$ are *separate* if $E_1 \cap E_2 = \emptyset$. 
In other words, the rule of sum can be seen as a specific case (i.e. \( E_1 \cap E_2 = \emptyset \)) of the familiar principle of inclusion and exclusion:

\[ |E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2| \]

3.3.2 Rule of product

**Example 1** How many permutations (arrangements) of \( n \) distinct terms \( x_1, x_2, \ldots, x_n \) are there in total?

The first element of the permutation can be any one of the \( n \) terms (\( n \) possibilities). The second element can be any one of the \( n \) elements except the term that was chosen as the first, giving \( n - 1 \) possibilities. The third element can be any one of the \( n \) elements except the terms that were chosen as the first two elements (\( n - 2 \) possibilities), and so on. Each element of the permutation, in order, corresponds to a separate event (think about why this is true), so that the total number of permutations is thus, by the rule of product, \( n \times (n-1) \times (n-2) \times \cdots \times 1 \), usually written \( n! \). The symbol “\( n! \)” is read “\( n \) factorial.”

For example, there are \( 2! = 2 \times 1 = 2 \) ways to arrange two distinct objects \( x_1 \) and \( x_2 \): either \( x_1, x_2 \) or \( x_2, x_1 \). There are \( 3! = 3 \times 2 \times 1 = 6 \) ways to arrange the three letters A, E, T into a “word”: the first letter can be either A, E, T (3 choices). Let us say that we picked A; then the second letter can be either E or T (2 choices). Let us say that we picked E; the the third letter can be only T (1 choice). Altogether, we can form the following six words: AET, ATE, EAT, ETA, TAE, TEA. There are \( 4! = 4 \times 3 \times 2 \times 1 = 24 \) orders in which the courses of a four course meal can be served.

**Example 2** How many ways are there to choose a subset of a set containing \( n \) elements?

We can regard the choice of a subset as a sequence of \( n \) decisions (events) whether or not to include each
of the $n$ elements in the subset. Each of these events can happen in two ways—an element is included or is rejected. Thus the rule of product tells us that number of different compound events, each event being the choice of a subset, can happen in $2 \times 2 \times \cdots \times 2 = 2^n$ ways.

For example, if we look at the set $\{H, E, L, P\}$. The first event is whether or not we include H (two possibilities). The second event is whether or not we include E (also two possibilities), and so forth.

* What is the number of rows in a truth table for a proposition having $n$ variables? Why? $+2$