Fifty three students took the exam; the statistics were:

- Minimum: 13
- Maximum: 88
- Median: 43
- Average: 45.85
- Std Dev: 17.12

1. Mathematical Induction.

Suppose we have a statement $S_n$ and we know the following facts:

(a) $S_1$ is true.

(b) If $S_n$ is true, then so is $S_{n-1}$.

(c) If $S_n$ is true, then so is $S_{2n}$

Prove by induction that $S_n$ is true for all integers $n \geq 1$.

First, (a) and (c) tell us (by induction) that $S_{2^k}$ is true for all $k \geq 0$. Now we prove by induction on $i$ that $S_{2^k-i}$ is true for all $k \geq 0$ and $0 \leq i < 2^k$. For $i = 0$, this follows because $S_{2^k}$ is true for all $k \geq 0$, as we just showed. If $i < 2^k - 1$ and $S_{2^k-i}$ is true, then by (b) $S_{2^k-(i+1)}$ is true, completing the induction.

2. Growth rates.

(a) Does $\binom{2n}{n}$ grow slower than, the same as, or faster than $n^n$? Prove your answer.

We calculate $\binom{2n}{n}$ using Stirling’s formula; ignoring lower order terms (which we never discussed):

$$\binom{2n}{n} = \frac{2n!}{(n!)^2} = \Theta \left( \frac{c\sqrt{2n(2n/e)^{2n}}}{(c\sqrt{n(n/e)^n})^2} \right) = \Theta \left( \frac{4^n}{\sqrt{n}} \right)$$

which grows much slower than $n^n$.

(b) Does $n(H_n)^2$ grow slower than, the same as, or faster than $n \log_2 n$? Prove your answer.

We showed in class that $H_n = \Theta(\log n)$ so that $n(H_n)^2 = \Theta(n \log^2 n)$ which grows faster than $n \log_2 n$.

3. Algorithms.

(a) Suppose you compute $2^n$ using the following recursive function:

```plaintext
FUNCTION PowerOfTwo(n)
BEGIN
    IF i=0 THEN RETURN 1;
    ELSE RETURN PowerOfTwo(n-1) + PowerOfTwo(n-1);
END
```
Analyze the number of additions needed to compute $2^n$.

$2^n - 1$ additions are used (by induction). The function does nothing more than add $1 + 1 + 1 + \cdots + 1$.

(b) Show how to compute $2^n$ in $O(n)$ additions.

Just avoid the duplicate call by using a temporary variable:

```plaintext
FUNCTION PowerOfTwo(n)
BEGIN
  IF i=0 THEN RETURN 1;
  ELSE temp := PowerOfTwo(n-1);
  RETURN temp + temp;
END
```

This does one addition per recursive call and there are $n - 1$ such calls (by induction).

(c) Using (b), analyze the number of additions to compute the sum $\sum_{i=0}^{k} 2^i$.

Each of the summands takes $O(i)$ from part (b), so together they take $\sum_{i=0}^{k} O(i) = O(\sum_{i=0}^{k} i) = O\left(\frac{k^2}{2}\right)$.


Use the rules of sum and product to give combinatorial interpretations of the identity

$$\left[\sum_{k=0}^{n} \binom{n}{k}\right]^2 = \sum_{k=0}^{2n} \binom{2n}{k} = 2^{2n} = 4^n.$$

With $n$ couples there are $2n$ people. By the Rule of Product, $2^{2n}$ is the number of subsets (of any size, including zero) of a set of the $2n$ people (each person can be in or out of the subset).

For each couple, we can have just the man, just the woman, both of them, or neither of them in the subset, four possibilities. There are $n$ couples, so the Rule of Product tells us there are $4^n$ ways to select the subset of the couples.

Such a subset can be formed by choosing $m$, the number of men in the subset, then that number of men, followed by choosing $w$, the number of women in the subset, then that number of women. Using the Rules of Sum and Product,

$$\left[\sum_{m=0}^{n} \binom{n}{m}\right] \left[\sum_{w=0}^{n} \binom{n}{w}\right] = \left[\sum_{k=0}^{n} \binom{n}{k}\right]^2.$$

Such a subset can also be formed by choosing $k$, the number of people in the subset, followed by choosing that number of people from the $2n$ men and women. Using the Rule of Sum this is

$$\sum_{k=0}^{2n} \binom{2n}{k}.$$

5. Binomial Coefficients.

Find the coefficient of $x^2$ in each of the following polynomials. You need not simplify powers, factorials, or binomial coefficients.
(a) \((1 + 2x)^{2013}\)

By the binomial theorem, \(2^2 \binom{2013}{2}\).

(b) \((1 - x)^{-2013}\)

By the binomial theorem extended to negative exponents, \(\frac{(-1)^{2014}}{2} \binom{2013 + 2 - 1}{2}\) = \(\binom{2014}{2}\).

(c) \((1 - x^3 + x^9 - x^{27} + \ldots)^{2013}\)

Zero, because the polynomial only has multiples of three powers of \(x\).

(d) \((1 - 2013x)^{2n+1}\)

By the binomial theorem, \(2013^2 \binom{2n+1}{2}\).