Illinois Institute of Technology
Department of Computer Science

Solutions to First Examination

CS 330 Discrete Structures
Spring Semester, 2008

1. **Mathematical Induction.**

   Prove by induction that for $n \geq 1$, $\sum_{k=1}^{n-1} k \times k! = n! - 1$.

   Basis step: for $n = 1, P(1) = 0 = 0$.

   Inductive step: assume $P(m)$ is true, show $P(m+1)$ is true. Let $P(m)$ be the statement $\sum_{k=1}^{m-1} k \times k! = m! - 1$. Then $P(m+1)$ is the statement $\sum_{k=1}^{m} k \times k! = (m+1)! - 1$. Rewrite LHS of $P(m+1)$ in terms of $P(m)$ by induction: $\sum_{k=1}^{m-1} k \times k! + m \times m! = m! - 1 + m \times m!$. But $m! - 1 + m \times m! = (m+1)! - 1$ and $P(m+1)$ follows.

2. **Growth rates.**

   (a) Is $\binom{n}{k} \in O(2^n)$ for $k \leq 10$? Prove your answer.

   $\binom{n}{k} = n^k/k! + O(n^{k-1})$ is a polynomial in n of degree k; in this case $k \leq 10$ is constant, and 2^n grows faster than any polynomial in n. If, say, $k = n/2$, the growth rate would be $n^{n/2}$ which grows faster than 2^n.

   (b) Is $nH_n \in O(n)$? Prove your answer.

   No. $H_n = \ln n + O(1)$ (as shown in class). Moreover, $n \times \ln n + k \leq C \times n$ would imply $\ln n + k \leq C$ which is false for all C as $n \to \infty$

3. **Algorithms/Binomial Coefficients.**

   (a) Suppose you compute $\binom{n}{i}$ with the recurrence

   $$\binom{n}{i} = \binom{n-1}{i-1} + \binom{n-1}{i}$$

   using the following recursive function:

```
FUNCTION Combination(n,i)
  BEGIN
    IF i=0 OR i=n THEN RETURN 1;
    ELSE RETURN Combination(n-1, i-1) + Combination(n-1, i);
  END
```

   Analyze the number of additions needed to compute $\binom{n}{i}$.

   If the recursion is expanded out, the computed is basically $1 + 1 + ... + 1$, so there are $\binom{n}{k} - 1$ additions. It also follows from the identity used, together with an induction that states "Computing $\binom{n}{i}$ by the above code takes $\binom{n}{i} - 1$ addition operations."

   (b) Show how to compute $\binom{n}{i}$ in $O(i)$ arithmetic operations.

   Simply write the binomial coefficient as the ratio of two products, $\binom{n}{i} = \frac{n(n-1)\cdots(n-i+1)}{i(i-1)\cdots 1}$. Evaluating the numerator takes i multiplications, as does the denominator, together with a single division.

(c) Using (b), analyze the number of arithmetic operations used to compute $\sum_{i=0}^{k}\binom{n}{i}$.

Each of the summands takes $O(i)$ from part (b), so together they take $\sum_{i=0}^{k} O(i) = O(\sum_{i=0}^{k} i) = O(k^2)$.

4. **More Binomial Coefficients.**

Find the coefficient of x^{31} in each of the following polynomials.

(a) $(1+x)^{25}$

We know from the bionomical theorem $(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$, so the x^j coefficient in $(1+x)^n$ is $\binom{n}{j}$ for $0 \le j \le n$, and 0 for $j < 0$ or $j > n$. Therefore the x^{31} coefficient for $(1+x)^{25}$ is 0.

(b) $(1-x)^{-25}$

We know from class that $(1-x)^{-n} = \sum_{k=0}^{\infty}\binom{n+k-1}{k}x^k$. Substituting $n = 25$, $(1-x)^{-25} = \sum_{k=0}^{\infty}\binom{25+k-1}{k}x^k$. So the x^{31} coefficient in $(1-x)^{-25}$ is $\binom{25+31-1}{31} = \binom{55}{31}$.

(c) $(1 - x^3 + x^6 - x^9 + \cdots)^5$

Because all powers of x in the expansion are multiples of 3, and 31 is not a multiple of 3, the answer is 0.

(d) $(1-3x)^{n+1}$

We know from the bionomial theorem $(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$. Substituting $-3x$ for x and $n+1$ for n in the binomial expression $(1+x)^n$ gives us $(1+(-3x))^{n+1} = \sum_{k=0}^{n+1}\binom{n+1}{k}(-3x)^k = \sum_{k=0}^{n+1}\binom{n+1}{k}(-3)^k x^k$ So the x^{31} coefficient in $(1-3x)^n$ is $(-3)^{31}\binom{n+1}{31}$.

5. **Combinatorial Interpretation.**

Give combinatorial interpretations of the identities

(a) $n! = \binom{n}{k}k!(n-k)!$

This is minor variation of the way combinations were counted with the variation of the rule of product.

LHS: Product rule for forming permutations of n elements. n ways to select first element, $n-1$ elements to select second element, $n-2$ ways to select third element, ...

RHS: Form a permutation of n elements by choosing a subset of size k of the n elements, then consider all $k!$ permutations of that subset as the first k elements of the permutation; then consider all $(n-k)!$ permutations of remaining $(n-k)$ elements.

(b) $\sum_{k\ge 0}\binom{n}{2k} = 2^{n-1}$

This is similar to one of the problems in HW 2.

LHS: Counts all even subsets of n.

RHS: You need a combinatorial way to get an even subset: remove one element from the set, call it X. Now choose any subset of the remaining $n-1$ elements; this can be done in 2^{n-1} ways by the rule of product. If the resulting subset has an even number of members, leave it alone; if it has an odd number of members, add X to it.