Illinois Institute of Technology
Department of Computer Science

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CS 330 Discrete Structures
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1 Another combinatorial identity

Let us examine the identity

$$\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r} \tag{1}$$

An algebraic proof is simple, but the combinatorial proof is more interesting. As before, we demonstrate that the combinatorial problem solved by the expression on the left-hand side of the equal sign and the combinatorial problem solved by the expression on the right-hand side of the equal sign are actually the same problem approached in two different fashions.

The left-hand side of (1) counts the number of possible outcomes of a two stage selection process of a set R of r elements from n. First, a subset $K \subseteq R$ of k elements is chosen and then r of these k are selected to form R. The rule of product says that this choice of R can happen in $\binom{n}{k}\binom{k}{r}$ ways. We can count the number of ways the same event can happen by first directly choosing r of the n elements as R (this can be done in $\binom{n}{r}$ ways) and then choosing from the other $n-r$ elements the remaining $k-r$ elements which when added to R form K (this can be done in $\binom{n-r}{k-r}$ ways). By the rule of product the compound event can occur in $\binom{n}{r}\binom{n-r}{k-r}$ ways. Since the compound event is the same in both of these applications of the rule of product, our proof is complete.

2 Vandermonde's identity

In the identities established so far, the algebraic proof has been very easy, almost eliminating the need for a combinatorial proof. We now present two identities for which the combinatorial proof is relatively simple and direct, but algebraic verification is not. Vandermonde's identity states that

$$\begin{aligned}\binom{n+m}{k} &= \sum_{i=0}^{k}\binom{n}{i}\binom{m}{k-i} \\ &= \binom{n}{0}\binom{m}{k} + \binom{n}{1}\binom{m}{k-1} + \cdots + \binom{n}{k}\binom{m}{0} \end{aligned} \tag{2}$$

The left-hand side of (2) is the number of ways to select a subcommittee of k people from a committee of n men and m women. On the other hand, the right-hand side counts the number of outcomes for the same problem: If the subcommittee is to have i men and $k-i$ women, the rule of product says that it can be chosen in $\binom{n}{i}\binom{m}{k-i}$ ways. By the rule of sum we must add this value for $i = 0, 1, \ldots, k$ to count the number of ways the subcommittee can be chosen. This proves Vandermonde's identity.

A similar identity states

$$\binom{n+m}{n} = \sum_{i=0}^{n}\binom{n}{i}\binom{m}{i}$$

$$= \binom{n}{0}\binom{m}{0} + \binom{n}{1}\binom{m}{1} + \cdots + \binom{n}{n}\binom{m}{n} \quad (3)$$

Note that, by our convention that $\binom{i}{k} = 0$ for $k > i$, if $n > m$, the last $n - m$ terms of the sum will be zero. Again, an algebraic proof is not as simple as a combinatorial one here. The left-hand side of (3) is the number of ways to select a subcommittee of n people from a committee of n men and m women. The selection of such a committee can also be done, however, by first choosing i, $0 \le i \le n$, to be the number of women on the subcommittee, choosing the i women in one of the $\binom{m}{i}$ possible ways, and finally choosing the $n - i$ men on the committee in one of the $\binom{n}{n-i} = \binom{n}{i}$ possible ways. The rules of sum and product give the right-hand side of (3).

When $n = m$, (3) becomes the interesting identity

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}.$$

3 The Binomial Theorem

We turn now to one of the most important applications of the binomial coefficients, indeed, the justification of that name for the values $\binom{n}{k}$. We begin by applying the combinatorial reasoning developed so far to a purely algebraic problem, the evaluation of $(1 + x)^n$. Writing down the first few values we find

$$\begin{aligned} (1+x)^0 &= 1 \\ (1+x)^1 &= 1 + x \\ (1+x)^2 &= 1 + 2x + x^2 \\ (1+x)^3 &= 1 + 3x + 3x^2 + x^3 \\ (1+x)^4 &= 1 + 4x + 6x^2 + 4x^3 + x^4 \\ &\vdots \end{aligned}$$

The coefficients of the powers of x on the right-hand sides of this equation are a reproduction of Pascal's triangle. Why? Using the algebraic rules of polynomial multiplication, we can reason as follows. Let

$$(1+x)^{n-1} = \binom{n-1}{0} + \binom{n-1}{1}x + \binom{n-1}{2}x^2 + \cdots + \binom{n-1}{n-1}x^{n-1}.$$

Multiplying this by $(1 + x)$ involves adding $(1 + x)^{n-1}$ and $x(1 + x)^{n-1}$:

$$\begin{aligned} (1+x)^{n-1} &= \binom{n-1}{0} + \binom{n-1}{1}x + \binom{n-1}{2}x^2 + \cdots + \binom{n-1}{n-1}x^{n-1} \\ x(1+x)^{n-1} &= \qquad\qquad \binom{n-1}{0}x + \binom{n-1}{1}x^2 + \cdots + \binom{n-1}{n-1}x^{n-1} + \binom{n-1}{n-1}x^n \end{aligned}$$

which gives

$$\begin{aligned} (1+x)^n &= \binom{n-1}{0} + (\binom{n-1}{1} + \binom{n-1}{0})x + (\binom{n-1}{2} + \binom{n-1}{1})x^2 + \cdots \\ &\quad + (\binom{n-1}{n-1} + \binom{n-1}{n-2})x^{n-1} + \binom{n-1}{n-1}x^n \end{aligned}$$

The coefficient of x^k in $(1+x)^n$ is found thereby to be the sum of the coefficients of x^{k-1} and x^k in $(1+x)^{n-1}$. Thus the computation of $(1+x)^n$ can be done using the *binomial theorem*:

$$\begin{aligned}(1+x)^n &= \sum_{k=0}^{n} \binom{n}{k} x^k \\ &= \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{n} x^n. \qquad (4)\end{aligned}$$

Equation (4) can be proven algebraically by induction, but, as before, we concentrate on a combinatorial argument. We ask what the coefficient of x^k is in the simplified product

$$(1+x)^n = \underbrace{(1+x)(1+x)\cdots(1+x)}_{n \text{ times}}$$

If we expand the product into the unsimplified sum of all the monomials (products of xs and 1s), we can then ask how many of those monomials will be x^k; that number will be the coefficient of x^k in $(1+x)^n$. We obtain the monomial x^k for each term in the unsimplified product formed by having k of the factors $(1+x)$ contribute an x and the other $n-k$ contribute a 1. Since there are n factors, this can happen in $\binom{n}{k}$ ways, verifying (4).

Equation (4) has many interesting consequences. For example, setting $x = 1$ gives

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n. \qquad (5)$$

yet another way. Setting $x = -1$ gives

$$0 = (1-1)^n = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots, \qquad (6)$$

The equation

$$\sum_{i=0}^{n} \binom{i}{k} = \binom{0}{k} + \binom{i}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1} \qquad (7)$$

can also be proved from (4) by a slightly more elaborate (and interesting) argument. We want to compute the value of

$$\sum_{i=0}^{n} \binom{i}{k}.$$

Now, $\binom{i}{k}$ is the coefficient x^k in $(1+x)^i$, so that the sum to be evaluated must be the coefficient of x^k in

$$\sum_{i=0}^{n} (1+x)^i = \frac{(1+x)^{n+1}-1}{(1+x)-1} = \frac{(1+x)^{n+1}-1}{x}$$

by the formula for the sum of a geometric progression.[1] This coefficient is the coefficient of x^{k+1} in $(1+x)^{n+1}-1$, which is $\binom{n+1}{k+1}$ by the binomial theorem, establishing (7).

[1] By long division of polynomials, $\frac{r^{n+1}-1}{r-1} = 1 + r + r^2 + \cdots + r^n$.

4 Applications of the binomial theorem

Equation (4) is useful when we are faced with the evaluation of a sum of binomial coefficients, because it allows us to transform such a sum into a sum of terms in a geometric progression.

Consider

$$\sum_{i=0}^{k} \binom{n+i}{i} \tag{8}$$

that we talked about before. We know that

$$\begin{aligned} \binom{n+i}{i} &= \text{coefficient of } x^i \text{ in } (1+x)^{n+i} \\ &= \text{coefficient of } x^n \text{ in } (1+x)^{n+i}x^{n-i}. \end{aligned}$$

Therefore,

$$\sum_{i=0}^{k} \binom{n+i}{i} = \text{coefficient of } x^n \text{ in } \textstyle\sum_{i=0}^{k}(1+x)^{n+i}x^{n-i}$$

and

$$\begin{aligned} \sum_{i=0}^{k}(1+x)^{n+i}x^{n-i} &= (1+x)^n x^n \sum_{i=0}^{k}(1+x)^i x^{-i} \\ &= (1+x)^n x^n \sum_{i=0}^{k}\left(\frac{1+x}{x}\right)^i \\ &= (1+x)^n x^n \frac{\left(\frac{1+x}{x}\right)^{k+1}-1}{\frac{1+x}{x}-1} \end{aligned}$$

Simplifying this polynomial, we have

$$\begin{aligned} &= (1+x)^n x^n \frac{\frac{(1+x)^{k+1}}{x^{k+1}}-1}{1/x} \\ &= (1+x)^n x^{n+1}\left[\frac{(1+x)^{k+1}-x^{k+1}}{x^{k+1}}\right] \\ &= (1+x)^n x^{n-k}\left[(1+x)^{k+1}-x^{k+1}\right] \\ &= (1+x)^{n+k+1}x^{n-k}-(1+x)^n x^{n+1}. \end{aligned}$$

Our sum is the coefficient of x^n in this sum. But $(1+x)^n x^{n+1}$ has no term x^n, so our sum is

$$\begin{aligned} \sum_{i=0}^{k} \binom{n+i}{i} &= \text{coefficient of } x^n \text{ in } (1+x)^{n+k+1}x^{n-k} \\ &= \text{coefficient of } x^k \text{ in } (1+x)^{n+k+1} \\ &= \binom{n+k+1}{k}, \end{aligned}$$

as we already knew. Other such summations require the more complicated manipulations, including the use of differentiation or integration.

5 More applications of the binomial theorem

As presented so far, the binomial theorem applies only to nonnegative integer powers of $(1+x)$. A much more general version can be derived by elementary calculus:

$$\begin{aligned}(1+x)^t &= 1 + tx + \frac{t(t-1)}{2!}x^2 + \frac{t(t-1)(t-2)}{3!}x^3 \\ &\quad + \cdots + \frac{t(t-1)(t-2)\cdots(t-k+1)}{k!}x^k + \cdots. \end{aligned} \tag{9}$$

This formula, which is the Taylor series expansion of $(1+x)^t$ around $x = 0$, is exactly equation (5) of last lecture, where t is a nonnegative integer. When t is negative or noninteger, the right-hand side of (9) is an infinite series that can be shown to converge for $|x| < 1$. Equation (9) suggests the generalization of the symbol $\binom{t}{k}$ to nonpositive or noninteger values of t:

$$\binom{t}{k} = \begin{cases} 1 & \text{if } k = 0, \\ t(t-1)(t-2)\cdots\frac{(t-k+1)}{k!} & \text{if } k > 0. \end{cases}$$

This allows us to write (9) more tersely as

$$(1+x)^t = \sum_{k=0}^{\infty} \binom{t}{k} x^k. \tag{10}$$

Notice that (10) includes (4) as a special case, since $\binom{n}{k} = 0$ for $k > n$ and integer n.

Of special interest is the case

$$(1-x)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} (-x)^k$$

where n is an integer.

By the definition of $\binom{t}{k}$, we have

$$\begin{aligned}\binom{-n}{k} &= \frac{(-n)(-n-1)(-n-2)\cdots(-n-k+1)}{k!} \\ &= \frac{n(n+1)(n+2)\cdots(n+k-1)}{k!}(-1)^k \\ &= \binom{n+k-1}{k}(-1)^k \end{aligned}$$

This gives

$$\begin{aligned}(1-x)^{-n} &= \sum_{k=0}^{\infty} \binom{n+k-1}{k} (-1)^k (-x)^k \\ &= \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k \end{aligned} \tag{11}$$

Equation (11) has a useful combinatorial interpretation. Rewriting $(1-x)^{-n} = (1+x+x^2+x^3+\cdots)^n$ by using the formula for the sum of a geometric progression, we obtain

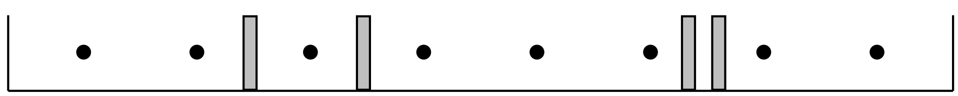
$$\underbrace{(1+x+x^2+\cdots)(1+x+x^2+\cdots)\cdots(1+x+x^2+\cdots)}_{n \text{ times}} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k$$

The coefficient of x^k on the left-hand side (which must, of course be the same as that on the right-hand side) is the number of ways to choose k objects from a set of n objects with unlimited repetition; that is, a single object can be chosen 0, 1, 2, ..., or k times. Why? Examine the way a term is formed in the unsimplified product on the left-hand side: it is a power of x from the first sum times a power of x from the second sum times a power of x from the third sum and so on. If x^k is to be formed in this way, the sum of the exponents in the powers of x must equal k. Such a product of powers $x^{i_1}x^{i_2}\cdots x^{i_n}$ equaling x^k thus corresponds to a selection of k objects as follows: i_1 of the first object, i_2 of the second object, ..., i_n of the nth object, for a total of $i_1+i_2+\cdots+i_n=k$ objects. It follows that the number of ways that x^k can appear in the unsimplified product is the number of choices of k objects from n objects with unlimited repetition. Equation (11) tells us that this is $\binom{n+k-1}{k}$.

Why is there an "$n-1$" in the above result? Let us think about it in another way. We want to know the number of ways to choose k objects from a set of n objects with unlimited repetition. Let $k=8$ and $n=5$. Suppose we have balls in 5 different colors, namely red, green, yellow, blue, and cyan, each with unlimitted number. How many ways are there to choose 8 balls from them?

We can pick the balls like this: first, we put 8 buttons in a long box.

Then, we insert $n-1=4$ boundary boards into the long box, which will seperate the long box into 5 smaller boxes. If there are m bottons in the ith box, we will pick m balls with the ith color (Notice that m can be 0). In the case shown below, we will pick 2 red balls, 1 green ball, 3 yellow balls, no blue balls, and 2 cyan balls.



Since we only have one way to put the buttons in the long box, the number of ways to pick up the k balls should be equal to the number of ways to insert the $n-1$ boundary boards. The latter interprets into the number of ways to pick $n-1$ out of $n-1+k$ positions (Imagine that the long box has $n-1+k$ slots. In each of the slots we can put either a button or a board. We just need to choose $n-1$ of them to put the boards, or k of them to put the bottons). Therefore, the number of ways to pick the balls should be

$$\binom{n+k-1}{n-1}=\binom{n+k-1}{k}$$

Let's look at another example: how many outcomes are possible if m standard dice are rolled? Each die has 6 faces that could be up, so we are choosing m faces (one for each die) from 6 possibilities, with repetition. Our discussion above showed that there are

$$\binom{6+m-1}{m}=\binom{m+5}{m}=\binom{m+5}{5}$$

ways to make the choice. For one die, there are just the

$$\binom{1+5}{5}=\binom{6}{5}=6$$

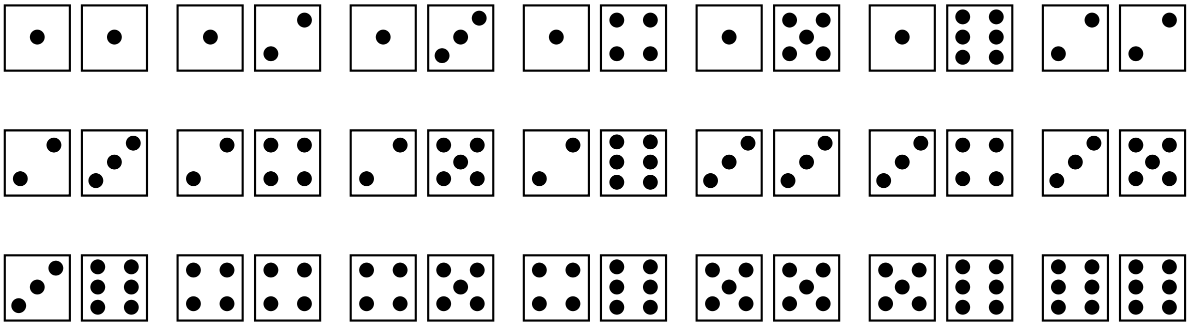


Figure 1: All possible pairs of dice.

obvious outcomes. For two dice there are the

$$\binom{2+5}{5} = \binom{7}{5} = \frac{7!}{5!2!} = 21$$

outcomes.

As a final example of the binomial theorem, we compute the coefficients in the binomial expansion of square roots:

$$(1+x)^{\frac{1}{2}} = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} x^k,$$

where

$$\binom{\frac{1}{2}}{k} = \begin{cases} 1 & k = 0, \\ \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\cdots(\frac{1}{2}-k+1)}{k!} & k = 1,2,3,\ldots \end{cases}$$

For $k = 1, 2, 3, \ldots$

$$\begin{aligned}
\binom{\frac{1}{2}}{k} &= \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{2k-3}{2}\right)}{k!} \\
&= \left(\frac{1}{2}\right)^k \frac{(-1)(-3)(-5)\cdots(-2k+3)}{k!} \\
&= (-1)^{k-1}\frac{1\times 3\times 5\cdots(2k-3)}{2^k k!} \\
&= (-1)^{k-1}\frac{1\times 2\times 3\times 4\times 5\cdots(2k-3)(2k-2)}{[2^k k!][2\times 4\times 6\cdots(2k-2)]} \\
&= (-1)^{k-1}\frac{(2k-2)!}{[2^k k!][2^{k-1}(k-1)!]} \\
&= \frac{(-1)^{k-1}}{2^{2k-1}}\frac{(2k-2)!}{k!(k-1)!} \\
&= \frac{(-1)^{k-1}}{2^{2k-1}}\frac{1}{k}\binom{2k-2}{k-1}.
\end{aligned}$$

Thus,

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{2^3}\frac{1}{2}\binom{2}{1}x^2 + \frac{1}{2^5}\frac{1}{3}\binom{4}{2}x^3 - \cdots. \tag{12}$$

Even this equation has an interesting combinatorial significance.