Logical Expressions

CS 350, Lecture 5, Wed Jan 25, 2012

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A. Why?

- Logical operations on bits is the lowest level of computation we do.
- Converting truth tables to expressions tells us how to do logical calculations.

B. Outcomes

At the end of today, you should:

- Be able to perform logical operations on individual bits using truth tables and simplifications.

C. Logical Values, Logical Operations/Connectives

- **Boolean logic** — named after George Boole
- The logical constants are true and false (typically written 1 and 0 or T and F).
- The logical operations / connectives are functions on 1 or 2 logical values.
    - Analogous to +, –, etc. on numbers.
    - NOT (unary), AND, OR, XOR, $NAND$, NOR (binary)
    - Others often used: $IMPL$, IFF
    - Truth tables are often used to show the possible results of logical expressions.

X	Y	X AND Y	X OR Y	X XOR Y	X NAND Y	X NOR Y	X IFF Y	X IMPL Y
0	0	0	0	0	1	1	1	1
0	1	0	1	1	1	0	0	1
1	0	0	1	1	1	0	0	0
1	1	1	1	0	0	0	1	1

X	NOT X
0	1
1	0

- $NOT\ X = 1$ iff $X = 0$
- $X\ AND\ Y = 1$ iff $X = Y = 1$
- $X\ OR\ Y = 1$ iff $X = 1$ or $Y = 1$ or both ("inclusive" or)
- $X\ XOR\ Y = 1$ iff $X \neq Y$ [one is 1, the other is 0] ("exclusive" or)
- $X\ NAND\ Y = NOT\,(X\ AND\ Y) = 0$ iff $X = Y = 1$

- X NOR $Y = NOT\,(X\ OR\ Y) = 1$ iff $X = Y = 0$
- More terminology:
  - $X\ AND\ Y$ is the **conjunction** of X and Y; its conjuncts are X and Y.
  - $X\ OR\ Y$ is the **disjunction**/**inclusive** OR of X and Y; its disjuncts are X and Y.
  - $X\ XOR\ Y$ is the **exclusive** OR of X and Y; its disjuncts are X and Y.
  - $IMPL$ is the **conditional** operator; IFF is the **biconditional**
    - If true and false are 1 and 0, then $IMPL$ behaves like $\le$.
      - $0 \le 0$, $0 \le 1$, $1 \le 1$, but not $1 \le 0$. I.e., $X\ IMPL\ Y$ behaves like $(NOT\ X)\ OR\ Y$.
- **Precedences** (higher/stronger to lower/lower): NOT, $AND/NAND$, OR/XOR, IMPL, IFF.
  - Example: $NOT\ X\ AND\ Y\ OR\ Z$ means $((NOT\ X)\ AND\ Y)\ OR\ Z$.
  - Example: $NOT\,(X\ AND\ NOT\ Y\ OR\ Y)$ means $NOT\,((X\ AND\ NOT\ Y)\ OR\ Y)$.
- **Associativity**:
  - For AND, OR, $NAND$, NOR, use **left associativity**.
  - For $IMPL$ and IFF, use **right associativity**.
- **Truth tables for larger expressions**
  - Number of rows in truth table = 2^n where n = number of variables used
  - For expressions with > 1 operator, there are two styles for truth tables.
  - Style 1: Each column contains a whole expression. E.g.

X	*Y*	*NOT Y*	*X AND NOT Y*	*X AND NOT Y OR Y*	*NOT (X AND NOT Y OR Y)*
0	0	1	0	0	1
0	1	0	0	0	1
1	0	1	1	1	0
1	1	0	0	1	0

  - Style 2: Columns can contain just the main operator of an expression.
    - Use extra parentheses to make it clearer what operands an operator has. E.g.:

X	*Y*	*NOT*	*((X*	*AND*	*NOT Y)*	*OR*	*Y)*
0	0	1	0	0	1	0	0
0	1	0	0	0	0	1	1
1	0	0	1	1	1	1	0
1	1	0	1	0	0	1	1

