

Tautologies; Normal Forms

CS 350, Lecture 6, Mon Jan 30, 2012

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A. Why?

- Tautologies describe properties that are always true and transformations that are always valid.
- Disjunctive Normal Form (DNF) is a particular standard way to represent a logical expression.
- DNF corresponds to a sum of the rows of 1's for the truth table of an expression.

B. Outcomes

At the end of today, you should:

- Know what tautologies, contradictions, and contingencies are and how to recognize them.
- Know what disjunctive normal form (DNF) is and how it's related to truth tables.

C. Tautologies, Contradictions, and Contingencies

- A logical expression is a
 - **Tautology** if it is always true (regardless of the values we assign its variables). I.e., its truth table has 1 in all rows.
 - **Contradiction** if it is always false (0's in all rows) .
 - **Contingency** if it has at least one row with a 1 and at least one row with a 0.
 - (I.e., it's neither a tautology nor a contradiction.)
 - If something is not a tautology, it's a contradiction or contingency
 - (Not always true) means (always false or sometimes false)
 - If something is not a contradiction, it's a tautology or contingency
 - (Not always false) means (always true or sometimes true)
- Two expressions are **equivalent** if their truth tables match.
 - If P and Q are equivalent, then $P \text{ IFF } Q$ is a tautology, and vice versa.
 - If P is a tautology, then $\text{NOT } P$ is a contradiction, and vice versa.
 - If P is a contingency, then $\text{NOT } P$ is also a contingency, and vice versa.

D. Manipulating Logical Expressions

- In addition to comparing the truth tables of expressions, we can also manipulate them syntactically, substituting equals for equals. ("Boolean" algebra.)

- Laws of Boolean Algebra (there exist many alternatives)
 - Associativity and Commutativity
 - *AND* and *OR* are associative; *XOR*, *NAND*, *NOR*, *IMPL*, *IFF* are not associative.
 - *AND*, *OR*, *XOR*, *NAND*, *NOR*, *IFF* are commutative, *IMPL* is not commutative.
 - Distributivity
 - $X \text{ AND } (Y \text{ OR } Z) = (X \text{ AND } Y) \text{ OR } (X \text{ AND } Z)$
 - $X \text{ OR } (Y \text{ AND } Z) = (X \text{ OR } Y) \text{ AND } (X \text{ OR } Z)$
 - DeMorgan's Laws
 - $P \text{ NAND } Q = \text{NOT } (P \text{ AND } Q) = (\text{NOT } P) \text{ OR } (\text{NOT } Q)$
 - $P \text{ NOR } Q = \text{NOT } (P \text{ OR } Q) = (\text{NOT } P) \text{ AND } (\text{NOT } Q)$
 - Contradiction: $X \text{ AND } \text{NOT } X = 0$
 - Excluded Middle: $X \text{ OR } \text{NOT } X = 1$
 - Double Negation (Pierce's Law): $\text{NOT } \text{NOT } X = X$
 - Identity: $X \text{ AND } 1 = X$; also, $X \text{ OR } 0 = X$
 - Domination: $X \text{ OR } 1 = 1$; also, $X \text{ AND } 0 = 0$
 - Idempotency: $X \text{ OR } X = X$ AND $X \text{ AND } X = X$
 - Definition of *IMPL*: $X \text{ IMPL } Y = \text{NOT } X \text{ OR } Y$
 - Definition of *IFF*: $X \text{ IFF } Y = (X \text{ IMPL } Y) \text{ AND } (Y \text{ IMPL } X)$

E. Alternate Notations

- Instead of words as operators, people often use symbols
 - For *IMPL*, *IFF*, can use \rightarrow (or \Rightarrow), \leftrightarrow (or \Leftrightarrow).
 - For *AND*, *OR*, can use \wedge , \vee .
 - For *AND* can use juxtapositioning or $*$
 - For *OR*, *XOR*, can use $+$, \oplus .
 - For *NOT*, can use \neg or \sim or overbar or prime: \bar{X} or X' .
- Distinguish $\bar{X} \bar{Y}$ (i.e., *NOT X AND NOT Y*) from $\overline{X Y}$ (i.e., *NOT(X AND Y)*).

F. Disjunctive Normal Form

- A **normal form** is a standard way of writing something.
- In DNF (Disjunctive Normal Form), a logical expression is written as as a disjunction (*OR*) of terms.
 - Each term is the conjunction (*AND*) of 1 or more "atoms" (a.k.a. "literals").
 - An atom/literal is a variable or *NOT* variable.

- **Example:** $\bar{X} Y + X Z$ is in DNF.
- **Example:** $X Z + Y Z$ is in DNF.
- **Example:** $(X+Y) Z$ is not in DNF (even though it's equivalent to $X Z + Y Z$).
- If an expression has k variables, then a DNF representation of it is in **full DNF** if each term has all k variables.
 - **Example:** $X Y \bar{Z} + \bar{X} Z$ is not in full DNF because the second term only has two variables. The equivalent full DNF representation is $X Y \bar{Z} + \bar{X} Y Z + \bar{X} \bar{Y} Z$.
- Using Boolean algebra (especially DeMorgan's laws and distribution) you can always simplify a complex expression into one in DNF.
- **Example:**

$$\begin{aligned} & \neg (X \bar{Y} + Y) \\ &= \neg (X \bar{Y}) \bar{Y} && \text{by DeMorgan's law (NOT of OR)} \\ &= (\bar{X} + Y) \bar{Y} && \text{by DeMorgan's law (NOT of AND) and double negation} \\ &= \bar{X} \bar{Y} + Y \bar{Y} && \text{by distributivity (AND over OR)} \\ & \text{(and if you want to further simplify it...)} \\ &= \bar{X} \bar{Y} + 0 && \text{by contradiction} \\ &= \bar{X} \bar{Y} && \text{by domination} \end{aligned}$$

G. DNF and Truth Tables

- Given a truth table column for an expression, each row in that column corresponds to a potential situation in which the expression is true. For example, in the table below, if we look for the times when Z is true, we find $\bar{X} \bar{Y} \rightarrow Z$ and $X \bar{Y} \rightarrow Z$ and $X Y \rightarrow Z$.

X	Y	Z
0	0	1
0	1	0
1	0	1
1	1	1

- We can combine implications because $(p \rightarrow r)$ and $(q \rightarrow r)$ implies $(p + q \rightarrow r)$.
 - E.g., $\bar{X} \bar{Y} \rightarrow Z$ and $X \bar{Y} \rightarrow Z$ combine to give $\bar{X} \bar{Y} + X \bar{Y} \rightarrow Z$.
 - We don't get an equality unless we have *exactly* the rows for which the expression is true; above, $Z = \bar{X} \bar{Y} + X \bar{Y} + X Y$.
- If you think of the truth table rows as being indexed by all the possible individual full DNF terms, then the expression is equivalent to the sum of those terms.

- Example: For $Z = \bar{X} \bar{Y} + X \bar{Y} + X Y$, we have

X	Y	<i>DNF Term</i>	Z
0	0	$\bar{X} \bar{Y}$	1
0	1	$\bar{X} Y$	0
1	0	$X \bar{Y}$	1
1	1	$X Y$	1

- For 3 variables, there are 8 rows, with each row indexed by a 3-variable term.

- Example: The table below illustrates $W = \bar{X} \bar{Y} \bar{Z} + \bar{X} Y Z + X \bar{Y} Z$.

X	Y	Z	<i>DNF Term</i>	W
0	0	0	$\bar{X} \bar{Y} \bar{Z}$	1
0	0	1	$\bar{X} \bar{Y} Z$	0
0	1	0	$\bar{X} Y \bar{Z}$	0
0	1	1	$\bar{X} Y Z$	1
1	0	0	$X \bar{Y} \bar{Z}$	0
1	0	1	$X \bar{Y} Z$	1
1	1	0	$X Y \bar{Z}$	0
1	1	1	$X Y Z$	0

- Not only can you translate a truth table to a full DNF representation, you can go the other way: Given a full DNF representation for an expression, you can give its truth table by inserting 1's into the positions whose indexes match up with DNF terms.

- Example: For $Z = \bar{X} \bar{Y} \bar{Z} + \bar{X} Y Z + X \bar{Y} Z$, we would insert three 1's, into the cells for $\bar{X} \bar{Y} \bar{Z}$, $\bar{X} Y Z$, and $X \bar{Y} Z$.