Simplifying Logical Expressions; Karnaugh Maps

CS 350, Lecture 7, Wed Feb 1, 2012

A. Why?

- Simplification of logical expressions can improve their readability and implementability.
- Karnaugh maps are one way to simplify logical expressions

B. Outcomes

At the end of today, you should:

- Know how to simplify boolean expressions using Karnaugh maps.

C. Using Truth Tables to Simplify Boolean Expressions

- Simplifying boolean expressions makes them easier to read and to implement.
- One way to simplify expressions uses factoring and other algebraic rules.
    - E.g., take the expression $\overline{X}\ \overline{Y} + X\,\overline{Y} + XY$.
    - We can factor X from the subterm $X\,\overline{Y} + XY$ and get $X(\overline{Y} + Y) = X\,1 = X$.
    - Similarly, $\overline{X}\ \overline{Y} + X\,\overline{Y} = (\overline{X} + X)\ \overline{Y} = 1\ \overline{Y} = \overline{Y}$.
    - Combining, we get

$$\begin{aligned}&\overline{X}\ \overline{Y} + X\,\overline{Y} + XY\\ &= (\overline{X}\ \overline{Y} + X\,\overline{Y}\,) + (X\,\overline{Y} + XY)\\ &= \overline{Y} + X\end{aligned}$$

    - (Note that we duplicated $X\,\overline{Y}$ so that we could use it in two factoring steps.)
- In terms of truth tables, factoring gives a way to simplify the sum of a clump of 1's.
    - In the table below, we again start with $\overline{X}\ \overline{Y} + X\,\overline{Y} + XY$. Each of the three terms contributes one row of 1's to the column.
    - To find a simpler equivalent expression, our technique will be to add terms that contribute 1's to the row until we have all the 1's we need.
    - Since the last two rows of the column correspond to the term X, we could take X and add $\overline{X}\ \overline{Y}$ to get the three 1's we need, but instead we can add $\overline{Y}$, which contributes the 1 at $\overline{X}\ \overline{Y}$ and re-contributes the 1 at $X\,\overline{Y}$. So we end with $X + \overline{Y}$.
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    -
    -

X	Y	$\overline{X}\,\overline{Y}+X\overline{Y}+XY$	$X+\ldots$	$\overline{Y}+\ldots$	$X+\overline{Y}+\ldots$	$X+\overline{Y}$
0	0	1	???	1	1	1
0	1	0	???	???	???	0
1	0	1	1	1	1	1
1	1	1	1	???	1	1

- If we let $Z = \overline{X}\,\overline{Y} + X\overline{Y} + XY$, then the process we followed was
  - $X \rightarrow Z$ (the rows where X is 1 are also rows where Z is 1)
  - $\overline{Y} \rightarrow Z$ (the rows where $\overline{Y}$ is 1 are also rows where Z is 1)
  - $Z \rightarrow X + \overline{Y}$ (the only rows where Z is 1 are the rows where X or $\overline{Y}$ are 1)
  - Hence $Z = X + \overline{Y}$.
- We'll say that if $P_0 + P_1 + \ldots + P_n$ implies Z, then each P_i is an **implicant** of Z. Then one way to simplify an expression is to reduce the number and size of implicants and making each implicant as "small" as possible. A **prime implicant** is an implicant where none of its conjuncts can be removed without changing the truth table.
  - **Example**: If $Z = \overline{X}\,\overline{Y} + X\overline{Y} + XY$, then $\overline{X}\,\overline{Y}$, $X\overline{Y}$, and XY are non-prime implicants of Z. The prime implicants of Z are X and $\overline{Y}$, and $Z = X + \overline{Y}$ is a best simplification we can get for Z.
- The implicants X and $\overline{X}$ are easy to search for because they correspond to two adjacent rows. Y and $\overline{Y}$ are harder because their two rows aren't adjacent.

D. Karnaugh Maps

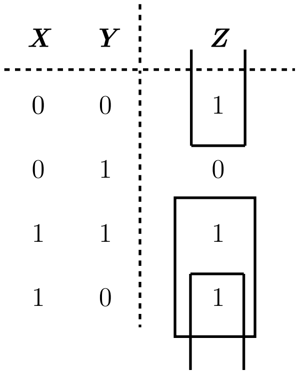
- **Karnaugh maps** ("car-no" maps or just **K-maps**) are a way to write the truth table for an expression in a format that makes it easier to find implicants for the expression and to recognize when the implicants are prime. The table below shows how adjacent rows of 1's can be combined to form a larger implicant. **It's very important to note that the rows are in the order 00, 01, 11, 10**, not 00, 01, 10, 11:

	X	Y	$X = XY + X\overline{Y}$	$Y = \overline{X}Y + XY$	$\overline{X} = \overline{X}\,\overline{Y} + \overline{X}Y$	$\overline{Y} = \overline{X}\,\overline{Y} + X\overline{Y}$
$\overline{X}\,\overline{Y}$	0	0	0	0	1	1
$\overline{X}\,Y$	0	1	0	1	1	0
$X\,Y$	1	1	1	1	0	0
$X\overline{Y}$	1	0	1	0	0	1

- By reordering the rows of the table, each adjacent pair of rows corresponds to a pattern of 0's and 1's where exactly one column is different; the two rows together correspond to a term where we drop the changing column. E.g., the last two rows correspond to $X\,Y + X\,\overline{Y}$; we can drop the Y and $\overline{Y}$ and get just X. The weird case is $\overline{Y}$, whose pair of rows wraps around the top and bottom of the table.
- For the example $Z = \overline{X}\ \overline{Y} + X\,\overline{Y} + XY$, reordering the rows gives us the table shown below. By looking for adjacent rows of 1's, we can see that X and $\overline{Y}$ are implicants and that $\overline{X}$ and Y aren't. We say that individually, X and $\overline{Y}$ both **partly cover** Z. Together, they **exactly cover** Z. I.e., $X + \overline{Y} = Z$.

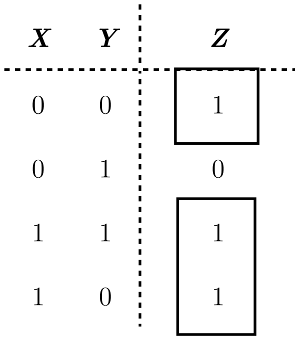
	X	Y	$\overline{X}\ \overline{Y}+X\,\overline{Y}+XY$	$X+\ldots$	$\overline{Y}+\ldots$	$X+\overline{Y}+\ldots$	$X+\overline{Y}$
$\overline{X}\ \overline{Y}$	0	0	1	???	1	1	1
$\overline{X}\ Y$	0	1	0	???	???	???	0
$X\ Y$	1	1	1	1	???	1	1
$X\,\overline{Y}$	1	0	1	1	1	1	1

- In a Karnaugh map, we indicate implicants by drawing a rectangle around the 1s for the implicant. For $\overline{Y}$, the rectangle is cut into two parts to indicate the wrap-around.



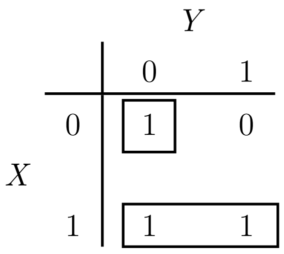
X	Y	Z
0	0	1
0	1	0
1	1	1
1	0	1

- An implicant is prime if it can't be extended to be a larger implicant. E.g., the map above for $X + \overline{Y}$ is preferred over the map below for $X + \overline{X}\ \overline{Y}$ because $\overline{X}\ \overline{Y}$ is not prime for the map below.

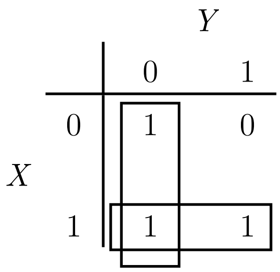


X	**Y**	**Z**
0	0	1
0	1	0
1	1	1
1	0	1

- In total, there are three kinds of implicants for a 2-variable map: Single squares, which correspond to a 2-variable conjunction; pairs, which correspond to a single variable, and the 4-high column of all 1's, which corresponds to T (true).
- A 2-variable Karnaugh map can also be drawn as a square: Below are maps for the expressions $X + \overline{X}\ \overline{Y}$ and $X + \overline{Y}$. (Again, $X + \overline{Y}$ is preferred because both its implicants are prime.)



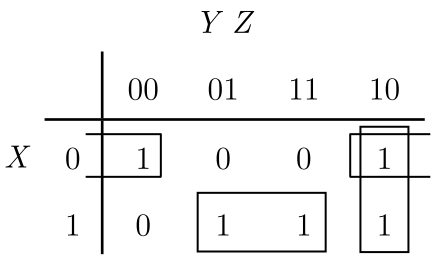
X \ Y	0	1
0	1	0
1	1	1



X \ Y	0	1
0	1	0
1	1	1

E. 3-Variable Karnaugh Maps

- A 3-variable Karnaugh map is basically two 2-variable maps joined together. In the example below, the top row is a map for Y and Z when $\overline{X}$ holds; the bottom row is a map for Y and Z when X holds.



X \ Y Z	00	01	11	10
0	1	0	0	1
1	0	1	1	1

- Since each row is a 2-variable map, we can look for pairs within each row:
  - Above, the top row includes $\overline{X}\,\overline{Z} = \overline{X}\,\overline{Y}\,\overline{Z} + \overline{X}\,Y\overline{Z}$ (the wrap-around pair)
  - The bottom row includes $X Z = X\overline{Y} Z + X Y Z$ (the middle pair)
- We can also look for pairs that go up and down; these correspond to simplifications that factor out $X + \overline{X}$, given some particular values for Y and Z.
  - In the example above, the rightmost column represents $Y\overline{Z} = \overline{X}\,Y\overline{Z} + X Y\overline{Z}$.

- With a 3-variable map, we can also look for 2 × 2 squares and 1 × 4 rectangles.
  - The table below left includes a square for $Z = \overline{X}\,\overline{Y} Z + \overline{X}\,Y Z + X\overline{Y} Z + X Y Z$.
  - Below right we have a square for $\overline{Z} = \overline{X}\,\overline{Y}\,\overline{Z} + \overline{X}\,Y\overline{Z} + X\overline{Y}\,\overline{Z} + X Y\overline{Z}$ (it wraps around left-to-right) and a rectangle for $X = X\overline{Y}\,\overline{Z} + X\overline{Y} Z + X Y Z + X Y\overline{Z}$.

		Y Z			
		00	01	11	10
X	0	0	1	1	0
	1	0	1	1	0

		Y Z			
		00	01	11	10
X	0	1	0	0	1
	1	1	1	1	1

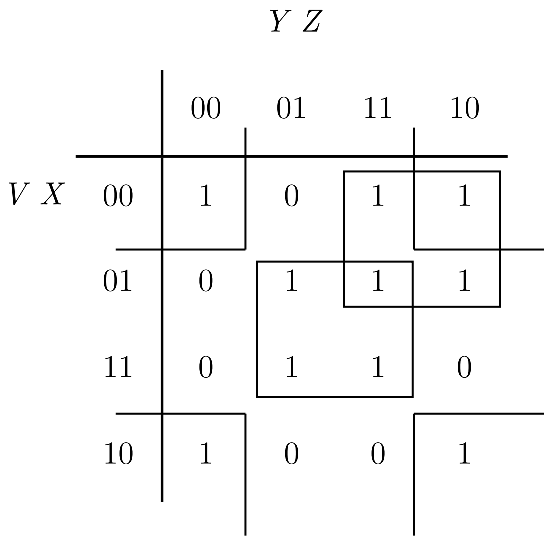
- A square or rectangle with 4 elements can also be seen as the combination of two pairs; attached side-to-side, the two pairs form a square; attached end-to-end, they form a rectangle.
- **No matter how many variables the map shows, implicants always contain numbers of elements that are powers of 2**. With a 3-variable map, the implicants contain 1, 2, 4, or 8 squares (the 8-element 2 × 4 rectangle of 1's represents T).
- A 3-variable map can also be written vertically, with each column representing a 2-variable map. The table below represents $X + \overline{Z}$, but this time, X is the square and $\overline{Z}$ is the 4 × 1 column.

		Z	
		0	1
X Y	00	1	
	01	1	
	11	1	1
	10	1	1

- A 3-variable map can also be written as a 2 × 2 × 2 cube where (for example) the top slice is a 2 × 2 table for $\overline{X}$ (and all combinations of Y and Z) and the bottom slice is a 2 × 2 table for X (and all combinations of Y and Z).

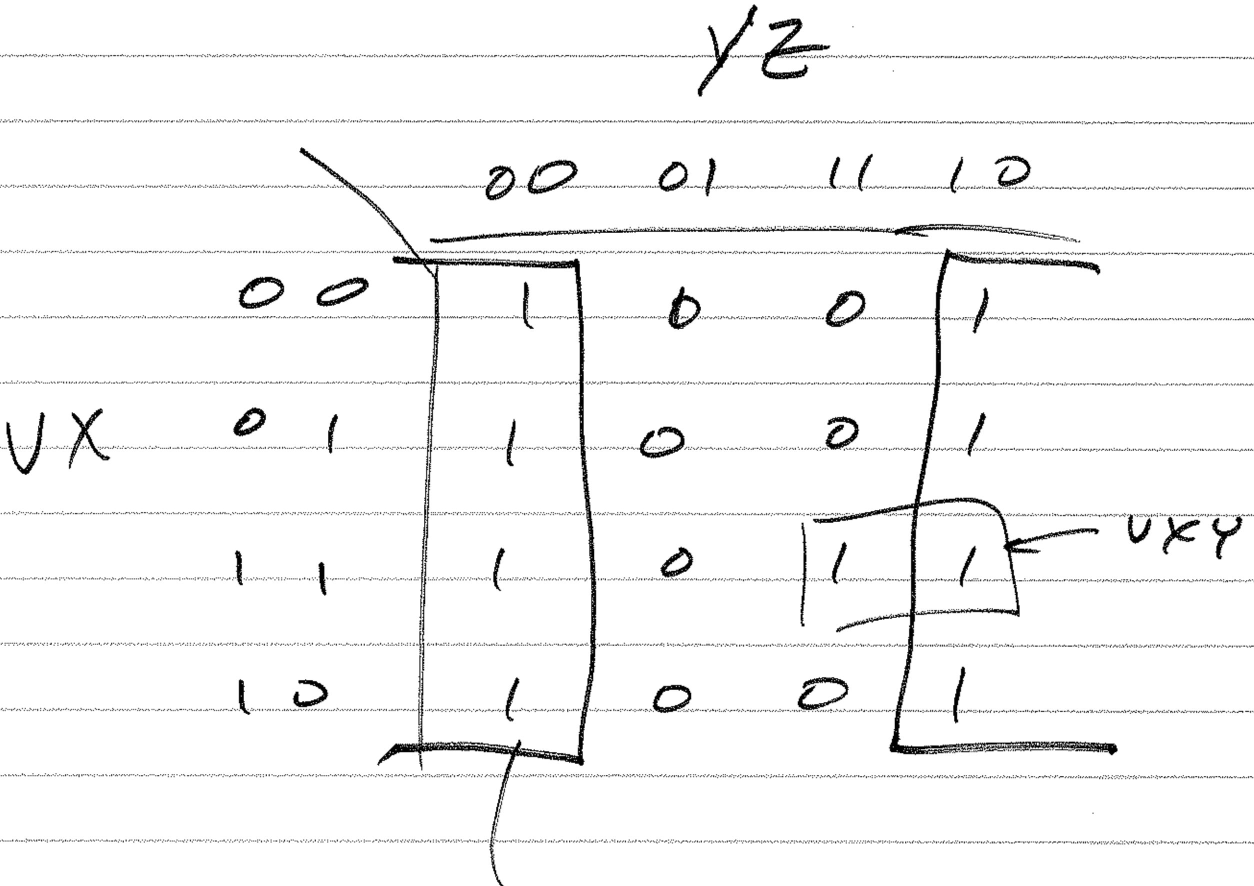
F. 4-Variable Karnaugh Maps

- For a 4-variable map, say for V, X, Y, and Z, we can use four 2-variable maps.
  - In the example below, each row is a 2-variable map for Y and Z, given some particular values for V and X.
  - The four rows are presented in $V\,X$ = 00, 01, 11, 10 order so that each column is a 2-variable map for V and X, given some particular values for Y and Z.
  - We can look for pairs, 2 × 2 squares, 4 × 1 and 1 × 4 rectangles, and 2 × 4 and 4 × 2 rectangles.



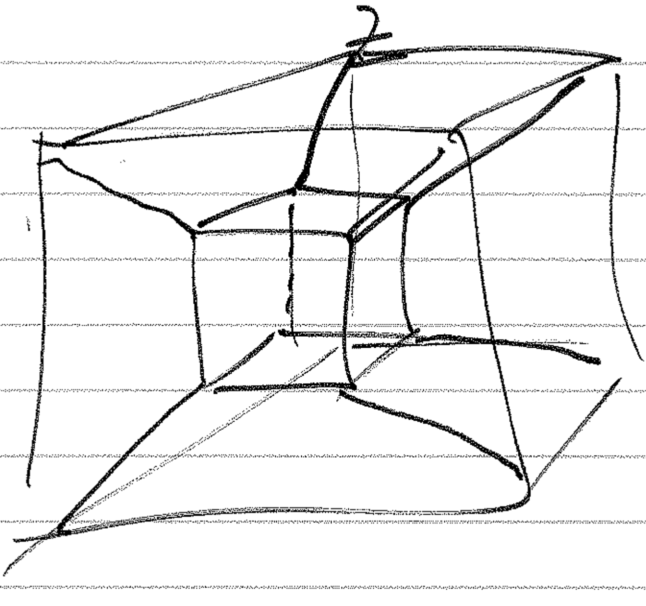
V X \ Y Z	00	01	11	10
00	1	0	1	1
01	0	1	1	1
11	0	1	1	0
10	1	0	0	1

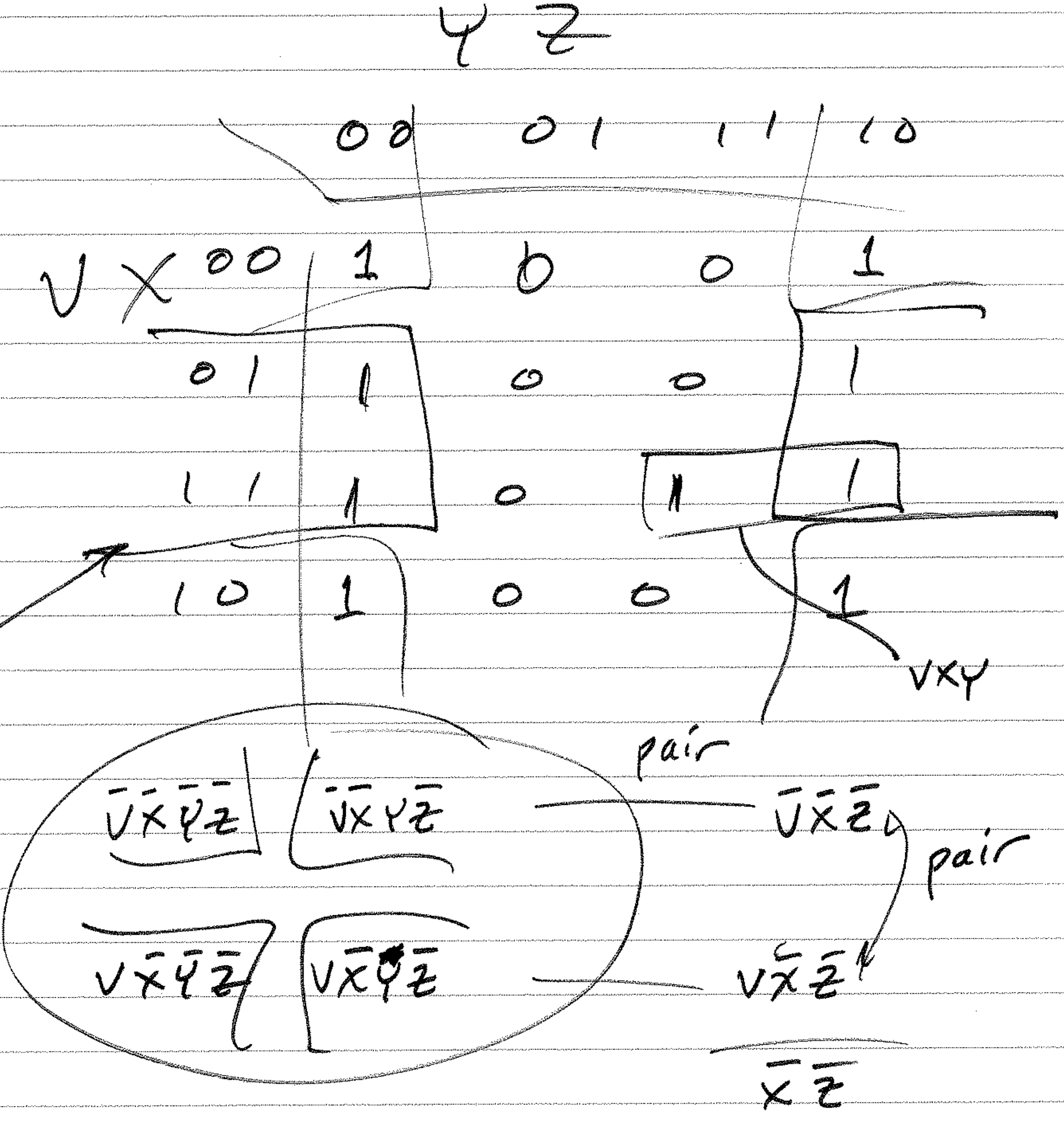
- The sample map above is for $X\,Z$ (the square in the middle) + $\overline{V}\,Y$ (the square at top right) + $\overline{X}\;\overline{Z}$ (the square that wraps around the four corners).



UX \ YZ	00	01	11	10
00	1	0	0	1
01	1	0	0	1
11	1	0	1	1
10	1	0	0	1

UXY





VX \ YZ	00	01	11	10
00	1	0	0	1
01	1	0	0	1
11	1	0	1	1
10	1	0	0	1

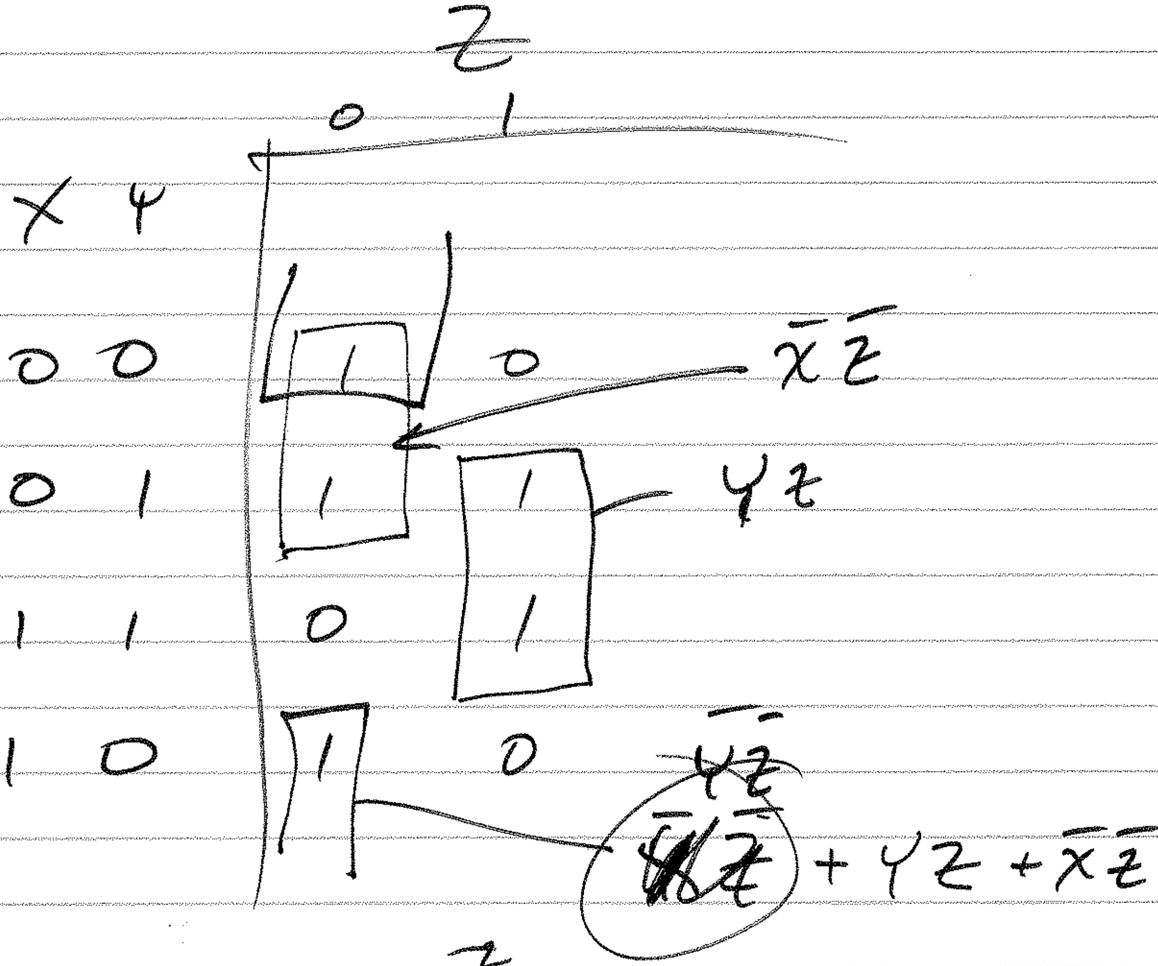
$\bar{V}\bar{X}\bar{Y}\bar{Z}$ $\bar{V}\bar{X}Y\bar{Z}$ — $\bar{V}\bar{X}\bar{Z}$

$V\bar{X}\bar{Y}\bar{Z}$ $V\bar{X}Y\bar{Z}$ — $V\bar{X}\bar{Z}$

$\bar{X}\bar{Z}$

$X\bar{Z} + VXY + \bar{X}\bar{Z}$

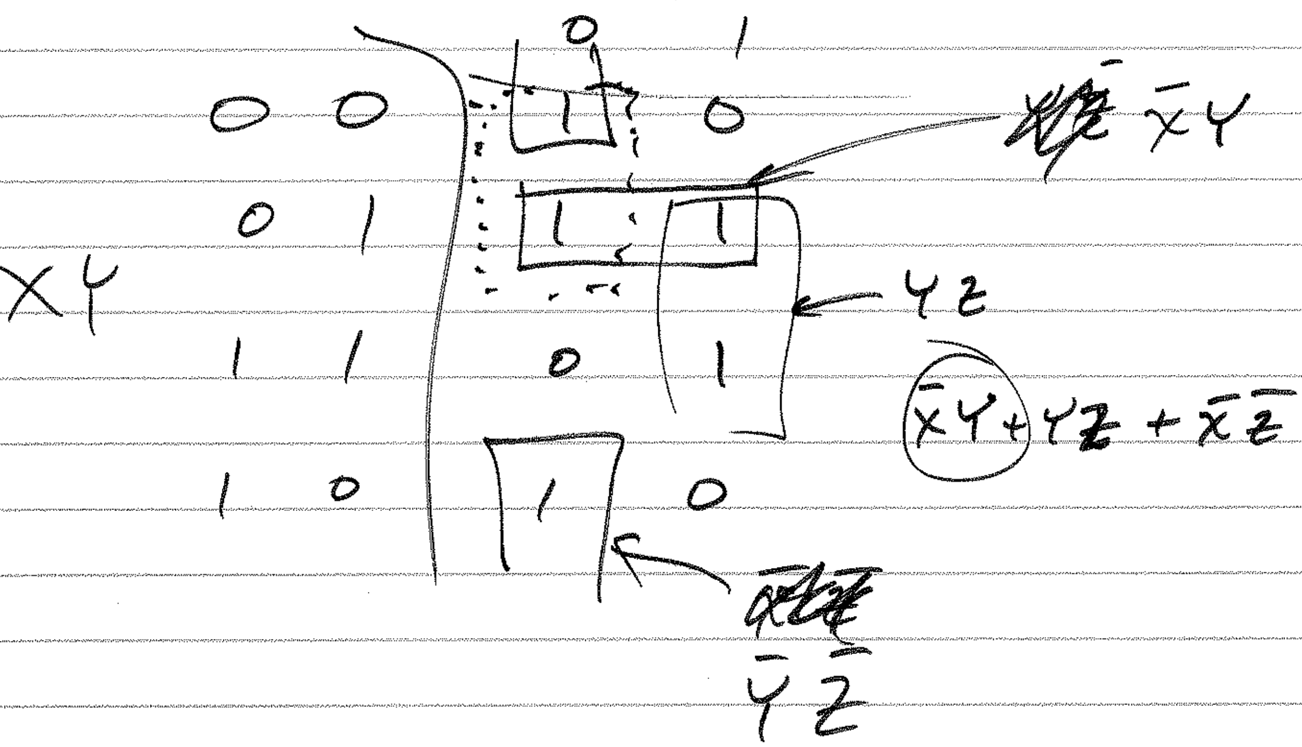
$Z + VXY$



XY \ Z	0	1
0 0	1	0
0 1	1	1
1 1	0	1
1 0	1	0

$\bar{Y}\bar{Z}$

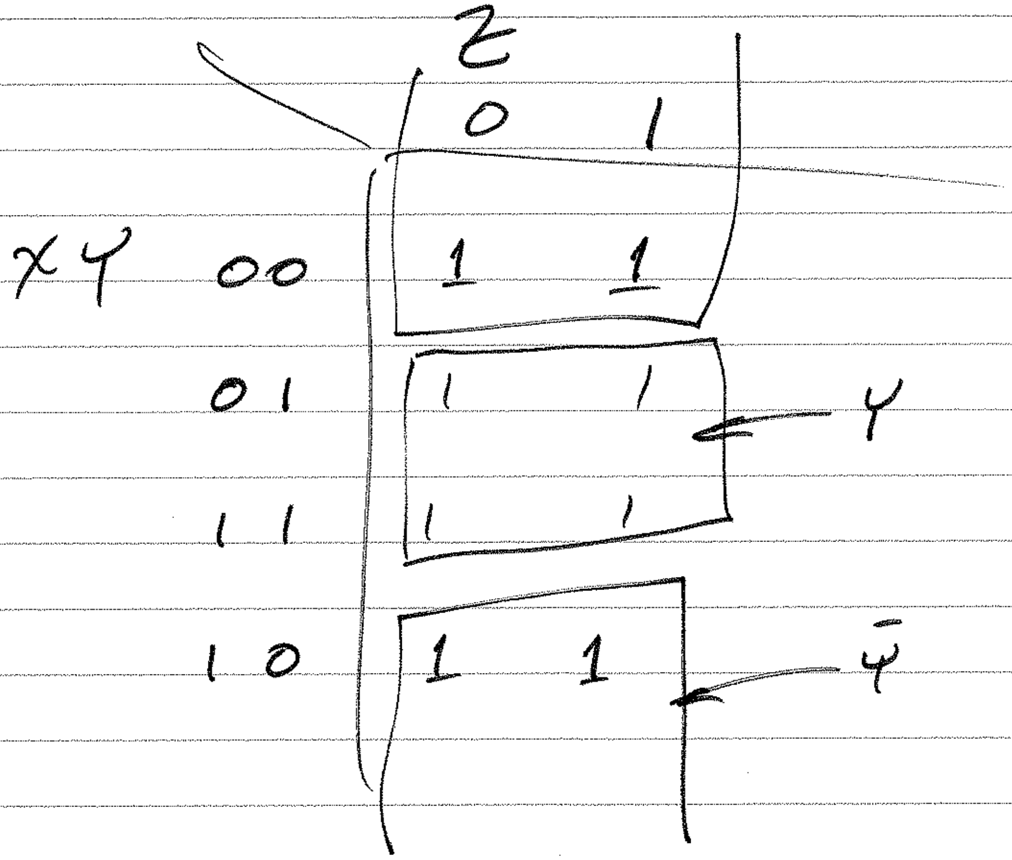
$\bar{Y}\bar{Z} + YZ + \bar{X}\bar{Z}$



XY \ Z	0	1
0 0	1	0
0 1	1	1
1 1	0	1
1 0	1	0

$\bar{X}Y + YZ + \bar{X}\bar{Z}$

$\bar{Y}\bar{Z}$



	Z	
XY	0	1
00	1	1
01	1	1
11	1	1
10	1	1

$\bar{x}\bar{Y}\bar{z} + \bar{x}\bar{Y}z$

$+ x\bar{Y}\bar{z} + x\bar{Y}Z$

$\bar{Y}\bar{Z} + \bar{Y}Z = \bar{Y}$

$\neg(xz) = \bar{x} + \bar{z}$

$\overline{xz} + vxyz + v\bar{x}\bar{z} + vx\bar{y}\bar{z}$

$vxy\bar{z}$

$\boxed{\neg(xz) + v(xyz + \bar{x}\bar{z}) + \cancel{\bar{x}x}}$

v x y z	expr
0 0 0 0	1
0 0 0 1	0
0 0 1 0	1
0 0 1 1	0
0 1 0 0	0
0 1 0 1	0
0 1 1 0	0
0 1 1 1	0
1 0 0 0	1
1 0 0 1	0
1 0 1 0 ~~1 0 0 0~~	1
1 0 1 1	0
1 1 0 0	1 oops
1 1 0 1	0
1 1 1 0	0
1 1 1 1	1

$$\begin{array}{c|cccc}
vx \backslash yz & 00 & 01 & 11 & 10 \\
\hline
00 & 1 & 0 & 0 & 1 \\
01 & 0 & 0 & 0 & 0 \\
11 & 1 & 0 & 1 & 0 \\
10 & 1 & 0 & 0 & 1
\end{array}$$

$\bar{x}\bar{z} + vxyz$

$+$

$v\bar{y}\bar{z}$

$(\bar{x})(\bar{z})$