1 Minimum Spanning Trees Definitions

$G = (V, E)$ is an undirected graph whose edges have weight $w$. A subgraph of $G$ is called spanning if it has $V$ as its vertex set. A spanning subgraph of $G$ is identified with its set of edges.

Let $A$ be a set of edges $A \subseteq E$. We say that edge $e$ is safe for $A$ if the following property holds: if $A$ is contained in some minimum spanning tree, then $A \cup \{e\}$ is contained in some minimum spanning tree.

A cut $(S, \bar{S})$ of a graph is a partition of $V$ into two nonempty sets $S$ and $\bar{S} = V \setminus S$.

We say that an edge $e$ crosses a cut $(S, \bar{S})$ if one of the endpoints of $e$ is in $S$ and the other endpoint is in $\bar{S}$.

We say that a cut respects a set of edges $A$ if no edge of $A$ crosses the cut.

An edge is a light edge crossing a cut if its weight is the minimum of any edge crossing the cut.

Theorem 1.1 Blue Rule. If the edge $e$ is light for some cut which respects the set of edges $A$, $e$ is safe for $A$.

Proof Sketch. (some details missing). Assume that there exists a minimum spanning tree $T$ (viewed as a set of edges) which contains $A$, that $(S, \bar{S})$ is a cut that respects $A$, and that $e$ is a minimum-weight edge crossing $(S, \bar{S})$. If $e \in T$, we are done, so let us assume $e \notin T$. Let $u$ and $v$ be the endpoints of $e$, with $u \in S$ and $v \in \bar{S}$. The unique path in $T$ from $u$ to $v$ must have an edge $e'$ such that $e'$ crosses $(S, \bar{S})$ and $e' \notin A$ since $(S, \bar{S})$ respects $A$.

We have that $T \setminus \{e'\} \cup \{e\}$ is another tree since it has the same number of edges and it is connected. And its weight is no more than the weight of $T$, since $e$ is a minimum-weight edge crossing $(S, \bar{S})$. Then $T \setminus \{e'\} \cup \{e\}$ is also a minimum spanning tree, and it contains $A \cup \{e\}$.