Print your name and student ID, neatly in the space provided below; print your name at the upper right corner of every page. Please print legibly.

Name:  
Student ID:  

This is an open book exam. You are permitted to use the textbook, any class handouts, anything posted on the course web page, any of your own assignments, and anything in your own handwriting. Foreign students may use a dictionary. Nothing else is permitted: No calculators, laptops, cell phones, Ipods, Ipads, etc.!

Do all four problems in this booklet. All problems are equally weighted, so do not spend too much time on any one question.

Show your work! You will not get partial credit if the grader cannot figure out how you arrived at your answer.

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1. **Augmented Red/Black Trees**

Recall the EPL($x$), the \textit{external path length} of the subtree rooted at a node $x$ in a red-black tree $T$, as in the lecture notes for January 31 (Lecture 7),

$$EPL(x) = \sum_{t \text{ is a leaf of subtree } x} \text{DEPTH}_x (l),$$

where \text{DEPTH}_x (l) is the depth of leaf $l$ in the subtree rooted at $x$. Let \text{SIZE}(x), the number of (internal) nodes in the subtree rooted at $x$. Consider a red-black tree augmented with a combination of both EPL($x$) and SIZE($x$). Can such a red-black tree be maintained without affecting the $O(\log n)$ performance of the insertion and deletion algorithms? Prove your answer.
2. **Dynamic Programming**

Various positive integers are arranged in an $N \times N$ triangular array $T$ such as

$$
\begin{array}{cccc}
(n=0) & 2 & & (k=1) \\
(n=1) & 5 & 4 & (k=2) \\
(n=2) & 3 & 4 & 7 & (k=3) \\
(n=3) & 1 & 6 & 9 & 6 & (k=4) \\
(n=4) & 3 & 4 & 10 & 5 & 2 & (k=5) \\
(n=5) & 2 & 4 & 8 & 16 & 3 & 1 \\
\end{array}
$$

We want to find the largest sum in a descent from the apex $(n=0, k=0)$ to the base $(n=5, k=5)$, in this case) through a sequence of adjacent numbers, one per each level, as shown by the circled values.

(a) Write a dynamic programming recurrence for the maximum sum.

(b) If you implement your recurrence *without memoization*, what is its running time (in terms of $N$)?

(c) If you memoize your implementation, what is its running time (in terms of $N$)?

(d) What additional memoization is needed to determine the *sequence of steps* used in to obtain the largest sum?
3. **Greedy Scheduling Variant**

Professor Reingold decided to sort the jobs in the activity-selection problem of section 16.1 in CLRS3 in *decreasing order by starting time* (instead of increasing order by finishing time).

(a) Give an example of a set of at least three jobs for which this greedy approach gives the best solution.

(b) Give an example of a set of at least three jobs for which this greedy approach does *not* give the best solution, or prove that no such example exists (that is, that this greedy heuristic always gives an optimum schedule).
4. Amortized Analysis

Consider the following modified version of the Increment code on page 454 of CLRS3:

1: \textbf{Increment}(A)
2: \hspace{1em} i \leftarrow 0
3: \hspace{1em} \textbf{while } i < A.\text{length} \text{ and } A[i] = 1 \text{ do}
4: \hspace{2em} A[i] \leftarrow 0
5: \hspace{2em} i \leftarrow i + 1
6: \hspace{1em} \textbf{end while}
7: \hspace{1em} \textbf{if } i < A.\text{length} \text{ then}
8: \hspace{2em} A[i] \leftarrow 1
9: \hspace{2em} \text{WasteTime}(i)
10: \hspace{1em} \textbf{end if}
11: \textbf{return}

The modification is line 9, where we have added the call to WasteTime(i). Let $T(i)$ be the worst-case running time of WasteTime(i). We know from the discussion in CLRS3 on pages 461–462 that Increment runs in amortized time $O(1)$ when $T(i) = 0$.

Use the potential function method paralleling the discussion on pages 461–462 to determine the amortized time of Increment if $T(i) = i$. 