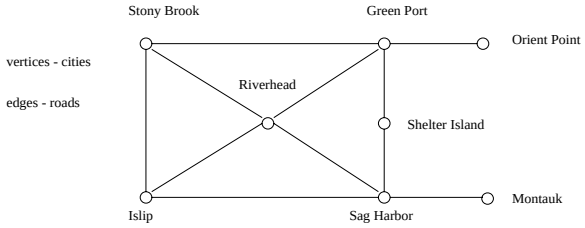
Graphs

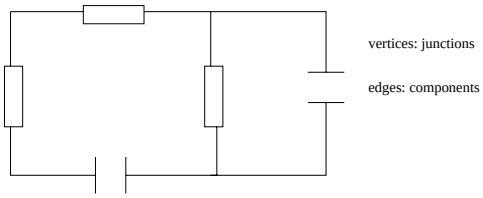
A graph G consists of a set of *vertices* V together with a set E of vertex pairs or *edges*.

Graphs are important because any binary relation is a graph, so graphs can be used to represent essentially *any* relationship.

Example: A network of roads, with cities as vertices and roads between cities as edges.



Example: An electronic circuit, with junctions as vertices as components as edges.



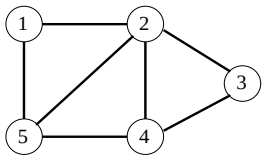
To understand many problems, we must think of them in terms of graphs!

Data Structures for Graphs

There are two main data structures used to represent graphs.

Adjacency Matrices

An *adjacency matrix* is an $n \times n$ matrix, where $M[i,j] =$ 0 iff there is no edge from vertex i to vertex j



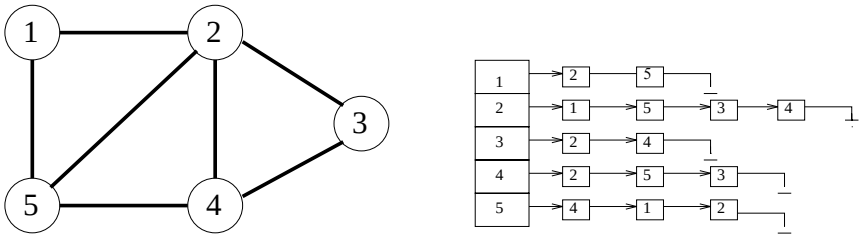
0	1	0	0	1
1	0	1	1	1
0	1	0	1	0
0	1	1	0	1
1	1	0	1	0

It takes $\Theta(1)$ time to test if (i,j) is in a graph represented by an adjacency matrix.

Can we save space if (1) the graph is undirected? (2) if the graph is sparse?

Adjacency Lists

An *adjacency list* consists of a $N \times 1$ array of pointers, where the ith element points to a linked list of the edges incident on vertex *i*.



To test if edge (i, j) is in the graph, we search the ith list for j, which takes $O(d_i)$, where d_i is the degree of the ith vertex.

Note that d_i can be much less than n when the graph is sparse. If necessary, the two *copies* of each edge can be linked by a pointer to facilitate deletions.

Tradeoffs Between Adjacency Lists and Adjacency Matrices

Comparison	Winner
Faster to test if (x, y) exists?	matrices
Faster to find vertex degree?	lists
Less memory on small graphs?	lists $(m + n)$ vs. (n^2)
Less memory on big graphs?	matrices (small win)
Edge insertion or deletion?	matrices $O(1)$
Faster to traverse the graph?	lists $m + n$ vs. n^2
Better for most problems?	lists

Both representations are very useful and have different properties, although adjacency lists are probably better for most problems.

Traversing a Graph

One of the most fundamental graph problems is to traverse every edge and vertex in a graph. Applications include:

- Printing out the contents of each edge and vertex.
- Counting the number of edges.
- Identifying connected components of a graph.

For *efficiency*, we must make sure we visit each edge at most twice.

For *correctness*, we must do the traversal in a systematic way so that we don't miss anything.

Since a maze is just a graph, such an algorithm must be powerful enough to enable us to get out of an arbitrary maze.

Marking Vertices

The idea in graph traversal is that we must mark each vertex when we first visit it, and keep track of what have not yet completely explored.

For each vertex, we can maintain two flags:

- *discovered* - have we ever encountered this vertex before?
- *completely-explored* - have we finished exploring this vertex yet?

We must also maintain a structure containing all the vertices we have discovered but not yet completely explored.

Initially, only a single start vertex is considered to be discovered.

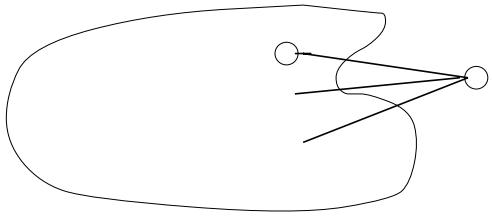
To completely explore a vertex, we look at each edge going out of it. For each edge which goes to an undiscovered vertex, we mark it *discovered* and add it to the list of work to do.

Note that regardless of what order we fetch the next vertex to explore, each edge is considered exactly twice, when each of its endpoints are explored.

Correctness of Graph Traversal

Every edge and vertex in the connected component is eventually visited.

Suppose not, ie. there exists a vertex which was unvisited whose neighbor *was* visited. This neighbor will eventually be explored so we *would* visit it:



Traversal Orders

The order we explore the vertices depends upon what kind of data structure is used:

- *Queue* – by storing the vertices in a first-in, first out (FIFO) queue, we explore the oldest unexplored vertices first. Thus our explorations radiate out slowly from the starting vertex, defining a so-called *breadth-first search*.

- *Stack* - by storing the vertices in a last-in, first-out (LIFO) stack, we explore the vertices by lurching along a path, constantly visiting a new neighbor if one is available, and backing up only if we are surrounded by previously discovered vertices. Thus our explorations quickly wander away from our starting point, defining a so-called *depth-first search*.

The three possible colors of each node reflect if it is unvisited (white), visited but unexplored (grey) or completely explored (black).

Breadth-First Search

BFS(G,s)
for each vertex $u \in V[G] - \{s\}$ do
    color[u] = white
    $d[u] = \infty$, ie. the distance from s
    $p[u] = NIL$, ie. the parent in the BFS tree
color[u] = grey
$d[s] = 0$
$p[s] = NIL$
$Q = \{s\}$
while $Q \neq \emptyset$ do
    $u = head[Q]$
    for each $v \in Adj[u]$ do
        if $color[v] = white$ then
            $color[v] = gray$
            $d[v] = d[u] + 1$
            $p[v] = u$
            enqueue[Q,v]
        dequeue[Q]
        $color[u] = black$

Depth-First Search

DFS has a neat recursive implementation which eliminates the need to explicitly use a stack.

Discovery and final times are sometimes a convenience to maintain.

DFS(G)
for each vertex $u \in V[G]$ do
    $color[u] = white$
    $parent[u] = nil$
$time = 0$
for each vertex $u \in V[G]$ do
    if $color[u] = white$ then DFS-VISIT[u]

Initialize each vertex in the main routine, then do a search from each connected component. BFS must also start from a vertex in each component to completely visit the graph.

DFS-VISIT[u]
$color[u] = grey$ (*u had been white/undiscovered*)
$discover[u] = time$
$time = time + 1$
for each $v \in Adj[u]$ do
    if $color[v] = white$ then
        $parent[v] = u$
        DFS-VISIT(v)
$color[u] = black$ (*now finished with u*)
$finish[u] = time$
$time = time + 1$