Problem 1.2-6: How can we modify almost any algorithm to have a good best-case running time?

To improve the best case, all we have to do it to be able to solve one instance of each size efficiently. We could modify our algorithm to first test whether the input is the special instance we know how to solve, and then output the canned answer.

For sorting, we can check if the values are already ordered, and if so output them. For the traveling salesman, we can check if the points lie on a line, and if so output the points in that order.

The supercomputer people pull this trick on the linpack benchmarks!

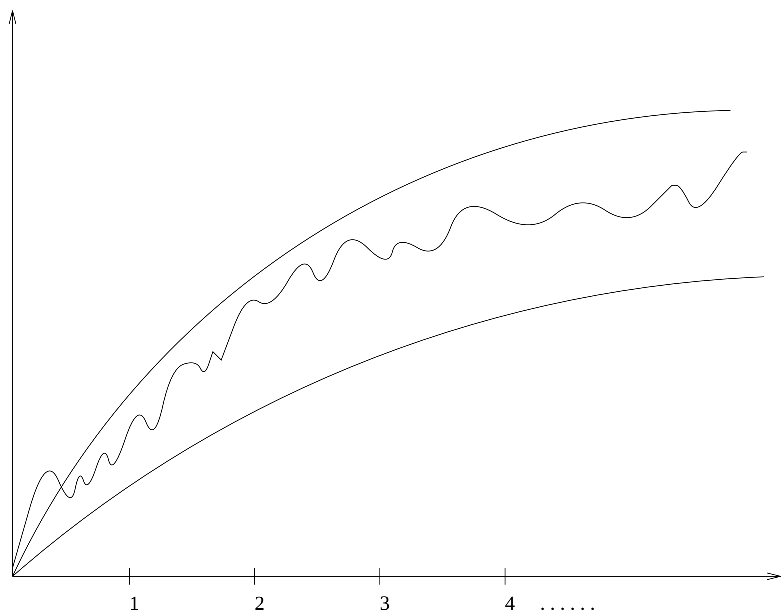
Because it is so easy to cheat with the best case running time, we usually don't rely too much about it.

Because it is usually very hard to compute the average running time, since we must somehow average over all the instances, we usually strive to analyze the worst case running time.

The worst case is usually fairly easy to analyze and often close to the average or real running time.

Exact Analysis is Hard!

We have agreed that the best, worst, and average case complexity of an algorithm is a numerical function of the size of the instances.



However, it is difficult to work with exactly because it is typically very complicated!

Thus it is usually cleaner and easier to talk about *upper and lower bounds* of the function.

This is where the dreaded big O notation comes in!

Since running our algorithm on a machine which is twice as fast will effect the running times by a multiplicative constant of 2 - we are going to have to ignore constant factors anyway.

Names of Bounding Functions

Now that we have clearly defined the complexity functions we are talking about, we can talk about upper and lower bounds on it:

- $g(n) = O(f(n))$ means $C \times f(n)$ is an *upper bound* on $g(n)$.

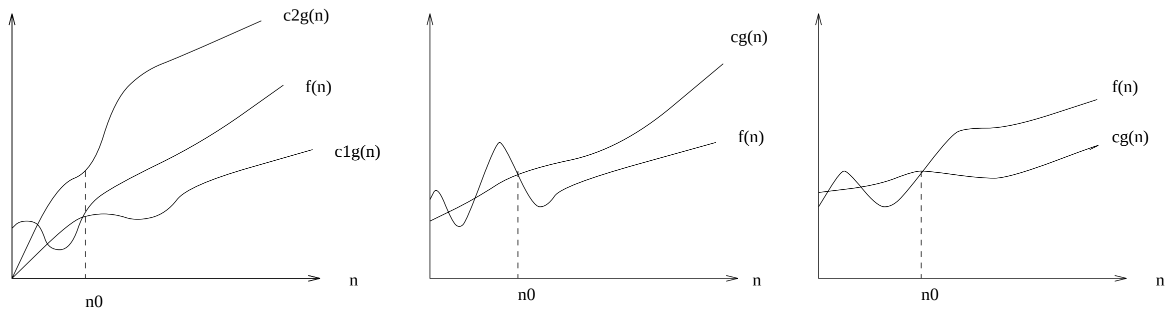
- $g(n) = \Omega(f(n))$ means $C \times f(n)$ is a *lower bound* on $g(n)$.

- $g(n) = \Theta(f(n))$ means $C_1 \times f(n)$ is an upper bound on $g(n)$ and $C_2 \times f(n)$ is a lower bound on $g(n)$.

Got it? C, C_1, and C_2 are all constants independent of n.

All of these definitions imply a constant n_0 *beyond which* they are satisfied. We do not care about small values of n.

O, Ω, and Θ



f(n) = O(g(n))

(a) (b) (c)

The value of n_0 shown is the minimum possible value; any greater value would also work.

(a) $f(n) = \Theta(g(n))$ if there exist positive constants n_0, c_1, and c_2 such that to the right of n_0, the value of $f(n)$ always lies between $c_1 \cdot g(n)$ and $c_2 \cdot g(n)$ inclusive.

(b) $f(n) = O(g(n))$ if there are positive constants n_0 and c such that to the right of n_0, the value of $f(n)$ always lies on or below $c \cdot g(n)$.

(c) $f(n) = \Omega(g(n))$ if there are positive constants n_0 and c such that to the right of n_0, the value of $f(n)$ always lies on or above $c \cdot g(n)$.

Asymptotic notation (O, Θ, Ω) are as well as we can practically deal with complexity functions.

What does all this mean?

$$
\begin{aligned}
3n^2 - 100n + 6 &= O(n^2) \; \textit{because} \; 3n^2 > 3n^2 - 100n + 6 \\
3n^2 - 100n + 6 &= O(n^3) \; \textit{because} \; .01n^3 > 3n^2 - 100n + 6 \\
3n^2 - 100n + 6 &\neq O(n) \; \textit{because} \; c \cdot n < 3n^2 \quad \textit{when} \quad n > c
\end{aligned}
$$

$$
\begin{aligned}
3n^2 - 100n + 6 &= \Omega(n^2) \; \textit{because} \; 2.99n^2 < 3n^2 - 100n + 6 \\
3n^2 - 100n + 6 &\neq \Omega(n^3) \; \textit{because} \; 3n^2 - 100n + 6 < n^3 \\
3n^2 - 100n + 6 &= \Omega(n) \; \textit{because} \; 10^{10^{10}} n < 3n^2 - 100 + 6
\end{aligned}
$$

$$
\begin{aligned}
3n^2 - 100n + 6 &= \Theta(n^2) \; \textit{because} \; O \quad \textit{and} \quad \Omega \\
3n^2 - 100n + 6 &\neq \Theta(n^3) \; \textit{because} \; O \quad \textit{only} \\
3n^2 - 100n + 6 &\neq \Theta(n) \; \textit{because} \; \Omega \quad \textit{only}
\end{aligned}
$$

Think of the equality as meaning *in the set of functions*.

Note that time complexity is every bit as well defined a function as $\sin(x)$ or you bank account as a function of time.

Testing Dominance

$f(n)$ dominates $g(n)$ if $\lim_{n\to\infty} g(n)/f(n) = 0$, which is the same as saying $g(n) = o(f(n))$.

Note the little-oh – it means "grows strictly slower than".

Knowing the dominance relation between common functions is important because we want algorithms whose time complexity is as low as possible in the hierarchy. If $f(n)$ dominates $g(n)$, f is much larger (ie. slower) than g.

- n^a dominates n^b if $a > b$ since

$$\lim_{n\to\infty} n^b/n^a = n^{b-a} \to 0$$

- $n^a + o(n^a)$ doesn't dominate n^a since

$$\lim_{n\to\infty} n^a/(n^a + o(n^a)) \to 1$$

Complexity	10	20	30	40
n	0.00001 sec	0.00002 sec	0.00003 sec	0.00004 sec
n^2	0.0001 sec	0.0004 sec	0.0009 sec	0.016 sec
n^3	0.001 sec	0.008 sec	0.027 sec	0.064 sec
n^5	0.1 sec	3.2 sec	24.3 sec	1.7 min
2^n	0.001 sec	1.0 sec	17.9 min	12.7 days
3^n	0.59 sec	58 min	6.5 years	3855 cent