Potential Functions for Dynamic Tables

CS 430 Introduction to Algorithms
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This is an informal justification for CLRS’s seemingly bizarre choice of potential functions in Section 17.4, the amortized analysis of dynamic tables.

Suppose the table size is doubled when an item is to be inserted when the table $T$ is full. There are only two parameters of $T$ that might contribute to a sensible potential function, its size $s = \text{size}(T)$ and the number of elements $n = \text{num}(T)$ in the table. For no good reason other than simplicity, let’s assume that the potential function is a linear combination of these two parameters, that is,

$$\Phi(T) = as + bn,$$

for some constants $a$ and $b$. When the table is full $\alpha(T) = 1$, and $n = s$; we better have enough potential to cover the cost of reinstering all $n$ elements in a table of size $2s$. In other words,

$$\Phi(T) = as + bn = n,$$

or

$$an + bn = n,$$

so that

$$a + b = 1.$$

After doubling the table size and doing all the reinsertions, the table is only half full $\alpha(T) = 1/2$ and $n = s/2$; we won’t need to resize the table for a long time, so we have time to build up potential. That is we don’t need any potential so,

$$\Phi(T) = as + bn = 0,$$

or

$$\Phi(T) = as + b\frac{s}{2} = 0,$$

implying

$$a + \frac{b}{2} = 0.$$

Solving these two simultaneous equations for $a$ and $b$ we get $a = -1$ and $b = 2$ and we arrive at the potential function

$$\Phi(T) = 2n - s.$$

Now suppose the table size is halved if an item is to be eliminated when the table is exactly $1/4$ full. Using a similar analysis to the above, we note that when the table has just been halved, that is $n = s/2$, we need no potential for a long while so that as before

$$\Phi(T) = as + bn = 0,$$

implying

$$a + \frac{b}{2} = 0.$$
But when the (shrinking) table becomes one quarter full, that is \( n = s/4 \), we better have enough potential to reinsert those items in a smaller table. That is,

\[
\Phi(T) = as + bn = n,
\]
or

\[
\Phi(T) = as + b \frac{s}{4} = \frac{s}{4},
\]

implying

\[
a + \frac{b}{4} = \frac{1}{4}.
\]

Solving these two simultaneous equations for \( a \) and \( b \) we get \( a = 1/2 \) and \( b = -1 \), and we arrive at

\[
\Phi(T) = \frac{s}{2} - n.
\]

Thus we need a more complex potential function, one for when the table is growing and another for when it is shrinking. We use

\[
\Phi(T) = \begin{cases} 
2n - s, & \text{if } \alpha(T) \geq \frac{1}{2}, \\
\frac{s}{2} - n & \text{if } \alpha(T) < \frac{1}{2}.
\end{cases}
\]

Notice that the two branches agree that \( \Phi(T) = 0 \) at the crossover point of \( \alpha(T) = 1/2 \); this is critical because we can’t have potential suddenly appearing or disappearing!

We can get a better sense of what this potential function means by expressing it in terms of \( \alpha(T) \):

\[
\frac{\Phi(T)}{s} = \begin{cases} 
2\alpha(T) - 1, & \text{if } \alpha(T) \geq \frac{1}{2}, \\
\frac{1}{2} - \alpha(T) & \text{if } \alpha(T) < \frac{1}{2}.
\end{cases}
\]

Graphically,