Dynamic Table Slides

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A **dynamic table** is a table of variable size, where an **expansion** (or a **contraction**) is caused when the load factor has become larger (or smaller) than a fixed threshold.

Let the expansion threshold be 1 and the expansion rate be 2; that is, *the table size is doubled when an item is to be inserted when the table is full.*

Let the contraction threshold be $1/4$ and the contraction rate be $1/2$; that is, *the table size is halved when an item is to be eliminated when the table is exactly $1/4$ full.*
When these operations take place we create a new table and move all the elements from the old one to the new one.

Suppose that there are $n$ calls of insertion and deletion are made, what is the average cost of each operation?
If the size is kept the same the cost is $O(1)$.

If the size is doubled from $M$ to $2M$, the actual cost is $M + 1$. The time that it takes for the next table size change to occur is at least $M$ steps for doubling and at least $M/2$ steps for halving. So the actual cost can be spread over the next $M/2$ “normal” steps. This gives an amortized cost of $O(1)$.

If the size is halved from $M$ to $M/2$, the actual cost is $M/4$. The time that it takes for the next table size change to occur is at least $M/4$ steps for doubling and at least $M/8$ steps for halving. So the actual cost can be spread over the next $M/8$ steps to yield an amortized cost of $O(1)$.
For each $i$, $1 \leq i \leq n$, define $c_i$ to be the number of insertions and deletions that are executed at the $i$-th operation, and define

$$\Phi_i = \begin{cases} 
2\text{num}_i - \text{size}_i & \text{if } \alpha_i \geq \frac{1}{2}, \\
\frac{\text{size}_i}{2} - \text{num}_i & \text{if } \alpha_i < \frac{1}{2}, 
\end{cases}$$

Here size$_i$ is the table size, num$_i$ is the number of elements in the table, and $\alpha_i$ is the ratio num$_i$/size$_i$ after the $i$-th operation. Note that

- at time 0, the table is empty, so $\Phi_0 = 0$,
- for all $i$, $\Phi_i \geq 0$, and thus, $\Phi_n \geq \Phi_0$, and
- $\Phi_n \leq 2n - n = n$, so the contribution of the potential function to the amortized cost is at most 1.
Here \( m = \text{num}_{i-1} \) and \( s = \text{size}_{i-1} \)

(a) \( \alpha_{i-1} = 1 \): Here \( m = s \).

\[
\begin{array}{c|c|c|c|c}
  c_i & \Phi_i & \Phi_{i-1} & \hat{c}_i \\
  m + 1 & 2(m + 1) - 2s & 2m - s & 3
\end{array}
\]

(b) \( \frac{1}{2} \leq \alpha_{i-1} < 1 \):

\[
\begin{array}{c|c|c|c|c}
  c_i & \Phi_i & \Phi_{i-1} & \hat{c}_i \\
  1 & 2(m + 1) - s & 2m - s & 3
\end{array}
\]

(c) \( \alpha_i = \frac{1}{2} \): Here \( m + 1 = \frac{s}{2} \).

\[
\begin{array}{c|c|c|c|c}
  c_i & \Phi_i & \Phi_{i-1} & \hat{c}_i \\
  1 & 2(m + 1) - s & s/2 - m & 0
\end{array}
\]

(d) \( \alpha_i < \frac{1}{2} \):

\[
\begin{array}{c|c|c|c|c}
  c_i & \Phi_i & \Phi_{i-1} & \hat{c}_i \\
  1 & s/2 - m - 1 & s/2 - m & 0
\end{array}
\]

So the amortized cost of insertion is \( O(1) \).
\( \alpha_i \geq \frac{1}{2} \):

<table>
<thead>
<tr>
<th>( c_i )</th>
<th>( \Phi_i )</th>
<th>( \Phi_{i-1} )</th>
<th>( \hat{c}_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2(m - 1) - s</td>
<td>2m - s</td>
<td>-1</td>
</tr>
</tbody>
</table>

\( \alpha_{i-1} = \frac{1}{2} \): Here \( 2m = s \).

<table>
<thead>
<tr>
<th>( c_i )</th>
<th>( \Phi_i )</th>
<th>( \Phi_{i-1} )</th>
<th>( \hat{c}_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{s}{2} - (m - 1) )</td>
<td>2m - s</td>
<td>2</td>
</tr>
</tbody>
</table>

\( \frac{1}{4} < \alpha_{i-1} \leq \frac{1}{2} \):

<table>
<thead>
<tr>
<th>( c_i )</th>
<th>( \Phi_i )</th>
<th>( \Phi_{i-1} )</th>
<th>( \hat{c}_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( s/2 - (m - 1) )</td>
<td>( s/2 - m )</td>
<td>2</td>
</tr>
</tbody>
</table>

\( \alpha_{i-1} = \frac{1}{4} \): \( m = \frac{s}{4} \) and \( \alpha_i < \frac{1}{2} \).

<table>
<thead>
<tr>
<th>( c_i )</th>
<th>( \Phi_i )</th>
<th>( \Phi_{i-1} )</th>
<th>( \hat{c}_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m )</td>
<td>( s/4 - (m - 1) )</td>
<td>( s/2 - m )</td>
<td>1</td>
</tr>
</tbody>
</table>

So the amortized cost of deletion is \( O(1) \).