Graph-3-coloring

The graph-3-coloring problem is

**Input:** An undirected graph $G = (V, E)$.

**Output:** Is there a coloring

$$c : V \rightarrow \{\text{red}, \text{blue}, \text{green}\}$$

such that for every edge $e$ in $E$ the vertices joined by $e$ are not colored with the same color?
Graph-3-coloring $\leq_p$ 3-SAT

Construct a 2-or-3-SAT Boolean expression from $G$ as follows.

For each vertex $v_i$ include a subexpression

$$(R_i \lor B_i \lor G_i) \land (\overline{R_i} \land \overline{G_i}) \land (\overline{R_i} \land B_i) \land (\overline{B_i} \land \overline{G_i})$$

$$= (R_i \lor B_i \lor G_i) \land (\overline{R_i} \lor \overline{G_i}) \land (\overline{R_i} \lor \overline{B_i}) \land (\overline{B_i} \lor \overline{G_i})$$

For an edge $e$ connecting $v_i$ and $v_j$ include a subexpression

$$(\overline{R_i} \land \overline{R_j}) \land (\overline{G_i} \land \overline{G_j}) \land (\overline{B_i} \land \overline{B_j})$$

$$= (\overline{R_i} \lor \overline{R_j}) \land (\overline{G_i} \lor \overline{G_j}) \land (\overline{B_i} \lor \overline{B_j})$$

Replace each 2-literal term $(a \lor b)$ with $(a \lor b \lor p) \land (a \lor b \lor \overline{p})$ for a new variable $p$. 
3-SAT $\leq_p$ Graph-3-coloring

Construct a graph $G$ from the 3-SAT expression as shown by the following example:

$$(a \lor \neg b \lor c) \land (b \lor d \lor \neg e)$$
The “or” Gadget—True Inputs
The “or” Gadget—False Inputs
The “or” Gadget—Mixed Inputs
Graph-3-coloring of Planar Graphs

\( \leq_p \text{3-SAT} \)

The following crossover gadget can be used to prove that determining whether a planar graph is 3-colorable is an NP-complete problem: