Upper Bound - $\Theta(n \log n)$ worst case

Quadratic - $O(n^2)$

$o(n \log n)$ is a lower bound in runtime

Quadratic

- $O(n \log n)$ (left)
- $O(n \log n)$ (right)

Smallest height possible

- Must lead to a root

$
\text{leaves} = n!
$

N leaves, height h

- Lemma: $N \leq 2^h$

Proof by induction

- $h > \log N$

$h=0$ $N=1$

$h=1$ $N=2$

$h=2$ $N=3$

$N=4$

$\log n$

Formulas

Upper Summation Formula

$= \sum_{i=1}^{n} \log i$

$= n \log n$
Any case

\[ \text{ave distance from root to leaf} \]

\[ \frac{1}{N} \sum_{\text{leaves}} \text{depth}(l) \]

\[ \frac{\min}{N} \sum_{\text{leaves}} \text{depth}(l) \]

\[ = \frac{1}{N} \left( \min \left[ \sum_{\text{leaves}} \text{depth}(l) \right] \right) \]

Lemma: tree w/ min looks like

\[ EPL(T) = \sum_{\text{leaves in } T} \text{depth of leaf} \]

Lemma: \( \min EPL \) occurs in a tree w/ all leaves at bottom 2 levels
If by contradiction - suppose not!

So, \[ L > L + 1 \]

\[ 0 > L + 1 - L \]

\[
\text{EPL} = \text{EPL} - 2L + (L-1) + 2(L+1) - L
\]

\[
\frac{\text{MIN}}{\text{before}} = \text{EPL} - L + L + 1
\]

\[
< \text{EPL}_{\text{before}}
\]
\[ l = \log_2 N = \frac{1}{c+1} \]

\[ h = 2 - N \]

\[ k = \frac{N}{h} + 1 \]

\[ EPL = (\# \text{ leaves at level } 2) \times 2 + \frac{(\# \text{ leaves at level } 2^{(k+1)} \times (k+1))}{N-h} \]

**Lemma:** If \( k_1, k_2, \ldots, k_N \) are the depths of the leaves in a tree, then

\[ \sum_{i=1}^{N} 2^{-k_i} \leq 1 \]

**Proof:**

1) Induction

2) Water pouring

\[ \min_{cPL} = N \log N + O(n) \quad \text{when } n! \]

\[ \min_{cPL} \text{ average depth of a leaf } = \frac{1}{n} \min_{cPL} = \frac{\log N}{N} + O(1) \]
Splay Tree

Min Avg Depth
of a leaf

= \log N

= \log n!

= n \log n + \Theta(n)