Amortized Analysis

D.S. \[ o_1, o_2, o_3 \ldots o_n \]

operations

cost of the sequence

worst case cost of many \( o_i \): \( c \)

\[ \Theta(nc) = \text{NAIVE} \]

\[ \frac{1}{n} \sum_{i=1}^{n} \text{cost}(o_i) = \text{avg. cost of an operation} \]

worst case average cost

Stack:

- push \( O(1) \) time \( \text{cost}(\text{push}) = \Theta(n) \)
- pop \( \Theta(h) \) - removes type h elt
- \( \Theta(h) \) - result in \( \Theta(n) \) item on stack

\( \Theta(n) \Rightarrow \text{cost}(\text{pop}) = \Theta(n^2) \)

Actual Costs

Amortized Costs

[\[ \begin{array}{l}
\text{$2/push} \\
\text{$0/ pop \ or \ multipop} \\
\end{array} \]

Amortized Costs

\[ \sum \text{Amortized Costs} = \sum \text{Actual Costs} \]
POTENTIAL PEN METHOD

\[ \Phi(catch) = \frac{\# \text{ of items on the stack}}{2} \]

ACTUAL COST = AMORTIZATION COST + \(\Phi(\text{before}) - \Phi(\text{after})\)

LABOR CONTINUOUS PUMP

\[ \sum \text{Actual costs} = \sum \text{Amortization costs} + \sum [\Phi(s_{ij})] \]

\[ \sum [\Phi(s_i) - \Phi(s_{i+1})] \]

AMORT COST = ACTUAL COST + \(\Phi(\text{after}) - \Phi(\text{before})\)

PUSH $1

AMORT COST = $2

PUSH

POP $1

AMORT COST

POP

MULTIPOP $6 - k = 0
### 17.1 Aggregate analysis

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### Figure 17.2
An 8-bit binary counter as its value goes from 0 to 16 by a sequence of 16 INC operations. Bits that flip to achieve the next value are shaded. The running cost for flipping is shown at the right. Notice that the total cost is always less than twice the total number of INC operations.

Operations on an initially zero counter causes $A[1]$ to flip $\lceil n/2 \rceil$ times. Si bit $A[2]$ flips only every fourth time, or $\lceil n/4 \rceil$ times in a sequence of $n$ INCR operations. In general, for $i = 0, 1, \ldots, k - 1$, bit $A[i]$ flips $\lceil n/2^i \rceil$ times in a sequence of $n$ INCREMENT operations on an initially zero counter. For bit $A[i]$ does not exist, and so it cannot flip. The total number of flip sequence is thus

$$
\sum_{i=0}^{k-1} \left\lceil \frac{n}{2^i} \right\rceil < n \sum_{i=0}^{\infty} \frac{1}{2^i} = 2n.
$$

by equation (A.6). The worst-case time for a sequence of $n$ INCREMENT op on an initially zero counter is therefore $O(n)$. The average cost of each of the and therefore the amortized cost per operation, is $O(n)/n = O(1)$.
\[ \phi_{(current)} = \# \text{ of hit bits} \]

\[ \text{AMORTIZED COST} = \text{ACTUAL COST INCR} + \left( \frac{\# \text{ of hit bits}}{\text{after}} \right) - \left( \frac{\# \text{ of hit bits}}{\text{before}} \right) \]

\[ \Phi_0 = 0 \]

\[ \Phi_1 > 0 \]

\text{HASH TABLES}

\text{insertion}

\text{deletion}