1. An infinite set is called countable if its elements can be placed in a one-to-one and onto correspondence with natural numbers. The set of natural numbers, \{0, 1, 2, 3, \ldots\}, is denoted in the following by \(\mathcal{N}\). Which of the following sets are countable? Prove your answers.

(a) The set of all subsets of size 2 of \(\mathcal{N}\).
(b) The set of all finite subsets of \(\mathcal{N}\).
(c) The set of all sets \(\mathcal{N}_{a,b} = \{a \times i + b \mid i \in \mathcal{N}\}\), for \(a, b \in \mathcal{N}\).
(d) The set of all subsets of \(\mathcal{N}\).

2. Let \(C\) be a language over an alphabet \(\Sigma\) and assume the symbol “;” does not belong to \(\Sigma\). Prove that \(C\) is Turing-recognizable if and only if a decidable language \(D\) exists such that \(C = \{x \mid \exists y (x; y \in D)\}\). Here \(x; y\) denotes the concatenation of the word \(x\), the symbol “;”, and the word \(y\).

**Hint:** For one direction of the proof, you get to choose what \(y\) would represent. Have \(y\) represent the number of steps a Turing Machine takes to finish.

3. Show that for any language \(A\), \(A\) is decidable if and only if \(A \leq_m \{1^n0^{2n} \mid n \geq 0\}\). Show that for any language \(A\), \(A\) is Turing-recognizable if and only if \(A \leq_m A_{TM}\).

4. Give a reduction (preferably a mapping reduction) showing that the language \(OVERBOARD_{TM} = \{<M> \mid M\) is a TM and for no input \(M\) moves its head left when the head is on the leftmost tape cell\} is not decidable.