Homework 5 version 1.1

Assigned: Apr. 13

Due: Apr. 27

The penalty for being late is changed for this homework: it must be turned in before the final exam (May 1), with a 10% penalty. No further extensions are possible.

1. Before Arabic numerals are invented, ancient people use items, such as sticks or shells, to count: each item counts as one, and a number can be represented by the size of a pile of items. In other words, we can use a multi-set of 1’s or 0’s to represent a number, and the number equals the size of the multi-set. In Ancient Egypt there was a ritual of divining. Given piles of sticks collected from all families in the village and a golden jar with shells, the arch vizier tries to choose some piles of sticks from all piles then tries to match each stick from the chosen piles with one shell from the jar; if there is no shell and stick left alone, it is a sign of prosperity. We can formulate the problem into a language, EGYPT-STICK-SHELL-DIVINE, or ESSD, as follows: \( w \in ESSD \) if and only if \( w = 1^{x_1}_1 | 1^{x_2}_1 | \ldots | 1^{x_k}_1 | 0^c \) for some sets \( X_1, X_2, \ldots, X_k \) and \( C \) and \( \exists \{a_1, a_2, \ldots, a_l\} \subseteq \{1, 2, 3, \ldots, k\} \) such that \( \Sigma_{i \in \{a_1, a_2, \ldots, a_l\}} |X_i| = |C| \). The alphabet of this language is \( \{0, 1, \&\} \).

Here, the notation \(|A|\) means the size of set \( A \). \( C \) is meant to be the set of shells and \( X_i \) is the set of sticks from the \( i^{th} \) family. Prove that \( ESSD \in P \). Feel free to use the RAM model (pseudocode as in the algorithms class). Argue correctness and polynomial running time.

2. In Ancient China, there was a type of lock that can be opened only by a special type of key. A lock consists of many round slots and pipes, such that each pipe connects a unique pair of slots. A key consists of many coins and stretchable ropes, such that each rope connects one unique pair of coins. A key can reorder its coins easily, since the ropes are soft and stretchable. If the coins can fit the slots and the ropes can fit the pipes perfectly after reordering the key, the lock will be opened. We can consider a lock and a key as graph \( L \) and \( K \) respectively: \( V(L) \) is the set of slots and \( E(L) \) is the set of pipes; \( V(K) \) is the set of coins and the \( E(K) \) is the set of ropes. We can formulate this problem into a language,

\[ \text{ANCIENT} - \text{CHINA} - \text{LOCK} = \{ \langle K \rangle, \langle L \rangle \mid K \text{ is identical to } L \text{ after reordering} \}. \]

Prove that \( \text{ANCIENT} - \text{CHINA} - \text{LOCK} \in NP \).

3. Let \( P^* = P \setminus \{\Sigma^*, \emptyset\} \), where \( P \) is the set of languages that are decidable in polynomial time on a deterministic single-tape Turing machine. Prove that if \( P = NP \), then every language \( L \in P^* \) is NP-complete.

4. The Set Partition problem is defined as follows. The input is a list \( S = s_1, s_2, \ldots, s_n \) of positive integers written in binary. The answer is “yes” if and only if there exists two sets \( A \) and \( B \) such that \( A \cup B = \{1, 2, \ldots, n\} \) and \( A \cap B = \emptyset \) and \( \Sigma_{s \in A} s = \Sigma_{s \in B} s \). As an example, if \( S = \{11, 32, 24, 9, 12\} \), one could pick \( A = \{1, 3, 4\} \) and \( B = \{2, 5\} \), so the answer is “yes”. For \( S = \{1, 3, 5\} \) the answer should be “no”.

Formulate Set Partition as a language Set-Partition, and prove that \( \text{CNF-SAT} \leq_p \text{Set-Partition} \). Easiest reduction seems to be from \( \text{SUBSET-SUM} \). If you use \( \text{SUBSET-SUM} \) or any other languages/decision problems for the reduction instead of \( \text{CNF} - \text{SAT} \), continue to argue using the notes that \( \text{CNF-SAT} \leq_p \text{Set-Partition} \).